Reinforced Concrete Slabs – Analysis and Design

An electronic textbook for basic concrete courses


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Examples

Scope

Dear reader

The purpose of this textbook is to introduce the basic understanding, the analysis and the design of reinforced concrete plates.

The book has been written with an engineering approach in mind: We will start by identifying the problems to be solved and then develop a method to solve these. We will also check that this leads to methods, which are valid according to the basic mechanical rules.

This inductive approach differs from the traditional deductive approach, but we will reach the same understanding, even if we often start with a simple problem, a simple test or a video of the testing and use this as a basis for the development.

I have uses this approach in my teaching of basic concrete structures at the Technical University of Denmark for few years and have found it easier for the students to understand - and often a lot more interesting for the students, as the relevance of other topics, assumptions and simplifications becomes much clearer.

I have produced this book as an electronic textbook for a number of reasons

1. The use of e.g. a video in a printed textbook is not possible, where the electronic form enables the book to be placed on a homepage with links to videos and pictures placed on YouTube, Picassa or other free and public sites.

2. The examples, exercises, solutions and in the end also the full classroom material will be made available for free download from the homepage in the latest versions, as eg. exercises are constantly updated, based on the feedback from students and assistant teachers.

3. The textbook can have a logical, linear part and at the same time have additional examples, videos, small tests and additional examples, which can be used for those of you, who prefer one or several additional examples.

4. The book can be updated whenever necessary, just as with additional examples and explanations can be added.

5. It is possible to distribute the book free of costs and thus also free of charge – as any of you can download it to you PC or print it.

The relevant material consists of this book, the examples, but also of supporting files and videos.

It need to be said, that at the book and the examples will be in **English** from the beginning, however, the exercises and the solutions and the rest will be in **Danish** at the moment, but these may be translated later.

You may try a translation machine on the Internet as eg. Google, which may provide a fair translation from Danish to English and enable you to understand the exercises and solutions – but the lecture videos will naturally require some understanding of Danish (sorry about that, but regulations requires these lectures to be in Danish as they deal with basic concrete structures).

The available material in English will thus be

- This textbook, organized in six main chapters (1)
- Relevant examples (1)

plus additional material in Danish

- Pdf copies of overheads for each chapter (1)
- Videos of each of the presentations of these chapters (2)
- Electronic examples as dynamic pdf-files for each chapter (1)
- Videos of the lecture, organised as one lecture per chapter (2)
- Exercises (1)
- Solutions to exercises (1)

The materials are available at

(1): The course homepage www.concretestructures.byg.dtu.dk

(2): The course account at Youtube, user: ConStruct2800Lyngby

What you need to know before you work with the plates: You must be able to analyse and design reinforced concrete beams, before you study the reinforced concrete plates and this means of course also that you must be able to estimate bending moments, shear forces and reactions.

You do not need to be familiar with plate theory or finite element methods, however, such knowledge will help you understand the models for the concrete plates and also enable you to go beyond the examples in the book. Additional knowledge of virtual work, lower limit and upper limit solutions will also be a help for your understanding, but are not vital.

I hope that you will find the book and the setup helpful for your understanding of reinforced concrete plates, their function, behaviour and design – and that you in due time will find the reinforced concrete as fascinating and useful as I do.

Enjoy

Per Goltermann
Professor
DTU-Byg.

…. and I would also like to thank all the other concrete fans at the university and in the industry, who have helped me with samples, photographs, videos – and perhaps those of my students, who kept asking until I actually came up with a simple and logical explanation, which they could understand easily. I have now tested the approaches on almost 1000 students from B.Sc. Civil Engineering, B.Eng. Building Engineering and B.Eng. Architectural Engineering - so I hope the book will have a fair chance of working with other students as well.

Symbols and definitions

A_s tensile reinforcement area;
b width of a beam;
c index for concrete;
D plate bending stiffness;
d effective height;
E_c modulus of elasticity of the concrete
$E_{c,long}$ long term modulus of elasticity for the concrete;
$E_{c,short}$ short term modulus of elasticity for the concrete;
E_s modulus of elasticity of reinforcement steel;
g dead load;
L span length, length of yield line;
i degree of restraint;
L_x, L_y lengths of the rectangular slab in the x and y directions;
M bending moment;
m bending moment per width;
m_i restraining moment per width along edge i;
m_n bending moment per width in the n direction
m_u bending moment capacity for positive bending;
m'_u bending moment capacity for negative bending;
m_x, m_y bending moment per width in the x and y directions;
m_{xy} torsion moment per width;
m_{xo},m_{yo},m_{xyo} maximal values of m_x, m_y and m_{xy} in a simply supported rectangular plate;
p distributed load;
p_x,p_y load transferred in the x and y directions;
$p_i^{(-)}$ lower limit solutions load carrying capacity for strip i;
q live load;
R concentrated reaction;
R_{ij} concentrated reaction at the junction of edge I and edge j;
r reaction per length;
s index for steel;
t thickness of the slab, coordinate along a line;
u deflection;
v shear force per width;
V shear force;
v_n shear force per width in the n direction;
v_x,v_y shear force per width in the x and y directions;
W_e,W_i external and internal work in failure mechanism for upper limit solution;
x,y coordinates and indexes for the x and y directions;
α ratio between modulus of elasticity of steel and concrete;
β, φ_b elastic parameters for estimation of elastic cross-sectional properties;
φ the creep factor;
ρ reinforcement degree.

1. Introduction

The best place to begin is to compare the beams and slabs (plate), which span in one direction only: There is no difference in the estimations of those two structural elements – but it will be different if we let the slabs span in two directions.

Figure 1.1. A few structures with single span prefabricated plates or slabs.

The design in practice is normally different as

- The function of a beam is normally to carry a load and this means that the widths of beams tend to be as small as possible, usually determined by the wish to place the tensile reinforcement at the bottom of the cross-section.

- The slab must be able to carry the load as well, but it will often be a lot wider, as the plates are normally used to fill a certain area, as eg. a separation of two floors in a building or to act as walls in building, exposed to wind loads.

This means that beams will normally have a high reinforcement degree in the range of 0,2-0,4, whereas slabs normally have lower degrees in the range of 0,05-0,1.

There are, however, limitations to the design of the single span slabs, which may be overcome by design and use of double span plates and use of plate theory.

1.1. Limitations to the single span slabs

The use of prefabricated slabs or plates spanning in one direction, is very popular in the constructions business, but has a serious limitation in their span width. This is best realised in the case of a slab with load q and eigenweight g and a span L and bending stiffness EI, where the maximal shear force V, bending moment M and deflection u are estimated as

$$V = \frac{1}{2}(g+q)L$$
$$M = \frac{1}{8}(g+q)L^2 \qquad (1.1)$$
$$u = \frac{5}{384}\frac{(g+q)}{EI}L^4$$

If we double the length of the span, then we will double V (factor 2) and quadrouble M (factor 4), but we will actually increase the deflection with a factor sixteen (factor 16) – this means that at larger spans, the deflection will determine the design or capacity of the slab as shown in Figure 1.2.

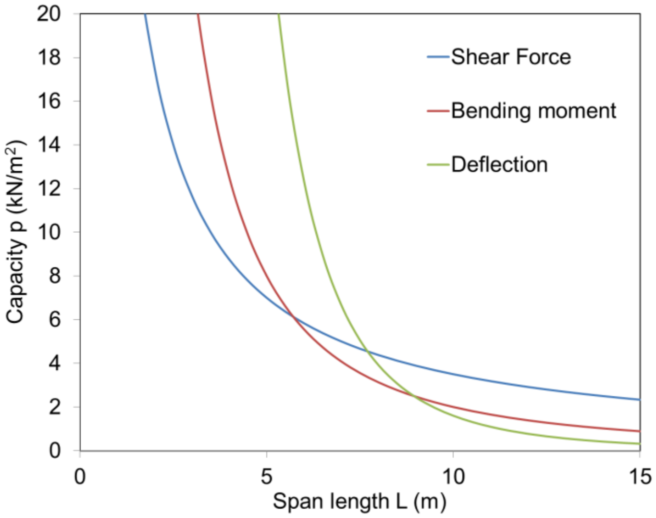


Figure 1.2. Capacity (p=g+q) in a slab with a specific cross-section as a function of the span L.

We may of course increase the reinforcement ratio (expensive and only possible up to a certain level) or we may increase the thickness of the slab (cheap), however, if we increase the thickness, then we increase the slabs own weight g equally. This means that longer spans require designs, where most of the slabs capacities are actually used for carrying the slabs own weight g and not the load q, which we need the slab to carry.

This would obviously be a poor and uneconomic design and we need to find a solution for this problem, if we wish to design longer spans.

A simple solution would be to let the slab span in two directions and use the strength of the concrete in two directions at the same time.

1.2. Double span slabs and their advantages

The simplest idea is really to place reinforcement in both directions and let the slab span in the two directions, so we can utilize the concrete in the compression zone in the two different directions, as this would allow us to carry the load in two different directions at the same time using the same concrete eigenweight and this would lead to a lower deflection, lower bending moments and lower shear forces.

Figure 1.2. Reinforcement arrangement for a double span bridge deck (left) and double spanning slabs in a parking garage at Illums Bolighus (right).

The effect can be demonstrated in the classroom: We place a slab with a single span on two opposite line-supports and load the slab with a load and measure the deflection. We repeat the test again with supports along all four sides, so the slab can span in both directions and we measure the deflection.

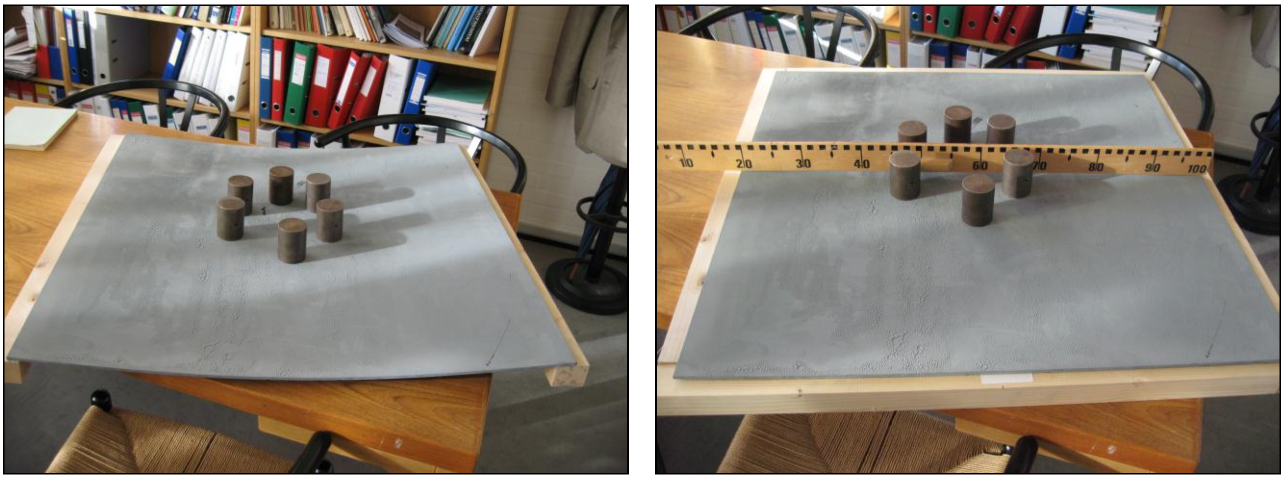


Figure 1.4. A isotropic slab with 6 kg load. Single span slab (left) and double span slab (right).
[Video of the demonstration available at Youtube]

The current slab was a plexiglass slab of 5 x 800 x 800 mm, loaded with 6 kg and we measured a maximal deflection at single span of 31 mm and at double span a deflection of only 10 mm.

We notice that the simple idea of describing the double spanning quadratic and isotropic slab as beam bending in the two directions would lead to predicted deflection of 50 % of single span slab. The deflection has, however, decreased to 10 mm / 31 mm = 32 %, which is much lower and much better – and we would like to benefit from this phenomenae.

Figure 1.5. The corner of the double spanning plate lifts up at the corner.

This simple test shows also that the correct behaviour of the slab is actually more complicated than just simple bending in two directions. A close look at the double span slab reveals that the corners of the slab actually lift from the supports during the test.

So this is an effect, which we can use to design slabs for much longer and more economical spans, but we need to get a better understanding of the way the double span slab works – and this is why we need the plate theory.

Notice: The term plate is used by the classic building mechanic theories for a plane specimens, where the thickness is significantly smaller than the width and length.

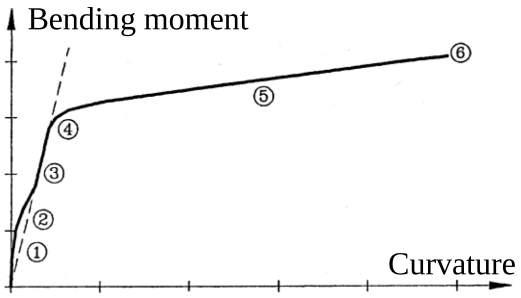
A concrete plate is normally termed “a slab” is concrete theories and literature, so:

“A SLAB IS A PLATE”

but we will in this book use the term “plate”, when we deal with the classic plate theory and the term “slab”, when we deal with concrete ~~plates~~ slabs.

1.3. What we need to be able to do

We need to take the non-linear behaviour of reinforced concrete in bending into account as we learned for the reinforced concrete beams.



Bending moment

1: Concrete is uncracked
2: Cracks start to form
3: Concrete is cracked
4: Reinforcement starts to yield
5: Reinforcement is yielding
6: Failure in concrete in the compression zone

Figure 1.5. Relationship between bending moment and curvature in a reinforced concrete beam.

The relationship may be tested but will normally deviate more or less from the theoretical relationship, as not all effects 1 to 6 are clearly visible in all tests.

We need also to be able to

- Determine the equilibrium and boundary conditions for slabs
- Determine the constitutive equations, which are the relationships between the generalized stresses and strains (as eg. bending moments versus curvatures)
- Determine the deflections in the Serviceability Limit State
- Determine or verify the loadcarrying capacity in the Ultimate Limit State

in order to design a double span concrete slab or to verify deflections and load-carrying capacities.

1.4. The rest of the book

The following topics will be dealt with in the chapters in the rest of the book

2. Classic slab theory (very brief)
3. Deflections in the uncracked and cracked states
4. Lower limit solutions with a guessed solution
5. Lower limit solutions with the strip method
6. Upper limit solutions with the yield line method

Each of these main chapters will be in their own chapter.

The chapter 2 on classic plate theory is very brief as many universities offers courses in plate theory and it is therefore mainly to list the formulas and relationships required for the chapter 3.

The chapter 3 deals with deformations in the serviceability limit state and utilize the plate theory.

The chapters 4 and 5 can be read independently of chapter 2 and 3, although the lower limit solutions must fulfil the equilibrium and boundary conditions, listed in chapter 2.

The chapter 6 can be read independently of the other chapters, although it is often interesting to compare the capacities derived by the methods in chapter 6 and 5.

1.5. Reading instructions for this book

The electronic text book use simple rules and logic combined with cases from tested specimens, examples and exercises and be organised in:

1. a linear part, which will be the student <u>must</u> understand, as it is the core of the method and which suitable for printing as one file and

2. key examples, which are <u>very important</u> for the understanding of the basic principles and assumptions and

3. a non-linear part, which includes <u>additional and optional</u> examples, exercises, videos, presentation of test cases, references to additional reading material, simple tests etc. This part is more dynamic, as additional cases, examples and exercises are added when required, since the number of examples needed for fully understanding the method and the possibilities will depend on the individual student and on the individual aspect.

The book has a brief English-Danish dictionary enclosed as the textbook is intended for use at Technical University of Denmark in Copenhagen, Denmark – where the lectures will normally be in Danish. (Teachers in other languages may likewise decide to add their own technical dictionary for their students).

2. Classic plate theory

We noticed in our simple test that the double spanning plate had some interesting advantages compared to the single spanning plate, but that it also had a more complex behaviour.

We will in this chapter (very briefly) outline the classic plate theory with equilibrium, boundary conditions and the generalised stress-strain relationships, so we can carry out the estimations later or set up simplified models and analysis.

The classic plate theory is based on the assumptions that

- The plate has a constant thickness
- The plate is thin, compared to its length and width
- The plate is exposed to pure bending
- The plane cross-sections remain plane and perpendicular to the centre plane
- The deflections are small compared to the thickness

which we will assume are also valid for normal concrete plates or slabs and we will use these assumptions to derive the plate theory in the following.

A more throughout introduction to plate theory can be found in the textbook from any university course covering plate theory and we will therefore keep the explanations as brief as possible and instead focus on the concrete slabs.

2.1. The classic plate theory

The generalised stresses in the plate theory are shown in Figure 2.1.

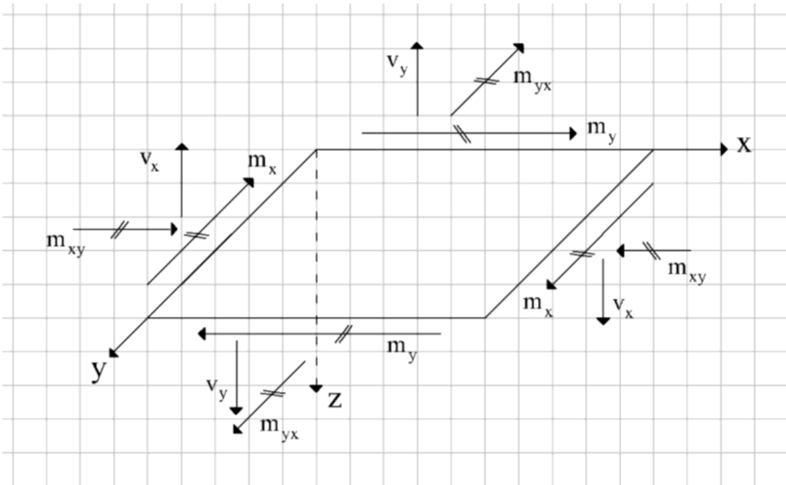


Figure 2.1. Generalised stresses

The generalised stresses are

m_x Bending moment per length, corresponding to bending of a fibre parallel to the x-axis
m_y Bending moment per length, corresponding to bending of a fibre parallel to y-axis
m_{xy} Torsional moment per length
v_x, v_y Shear force per length, corresponding to load transport along the x-axis and y-axis

These generalised stresses are derived from the stresses in the same way as for beams, except that they are per width and not for the whole width as for the beams.

$$
\begin{aligned}
m_x &= \int_{-h/2}^{h/2} \sigma_x z dz \\
m_y &= \int_{-h/2}^{h/2} \sigma_y z dz \\
m_{xy} &= \int_{-h/2}^{h/2} \tau_{xy} z dz = m_{yx} \\
v_x &= \int_{-h/2}^{h/2} \tau_{xz} dz \\
v_y &= \int_{-h/2}^{h/2} \tau_{yz} dz
\end{aligned}
\tag{2.1}
$$

The distributed load p and the resulting deflection u are positive in the same direction of the z-axis. The load p must be in equilibrium with the generalised stresses as shown in Figure 2.2.

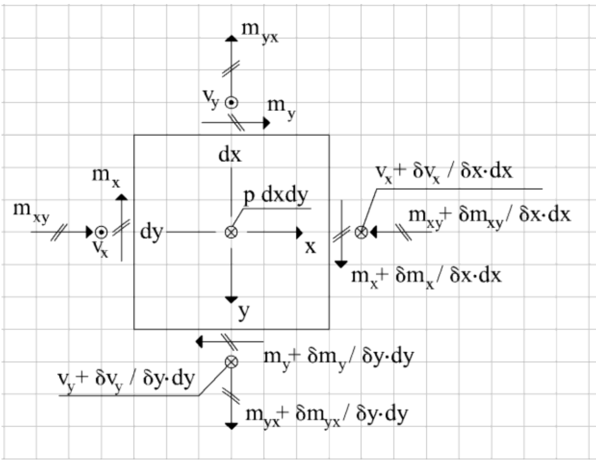


Figure 2.2. Distributed load and generalised stresses

The vertical equilibrium and the moment equilibriums around the x and the y-axes lead to the formulas

$$
\begin{aligned}
&\frac{\partial^2 m_x}{\partial x^2} + 2\frac{\partial^2 m_{xy}}{\partial x \partial y} + \frac{\partial^2 m_y}{\partial y^2} + p = 0 \\
&\frac{\partial m_x}{\partial x} + \frac{\partial m_{xy}}{\partial y} = v_x \\
&\frac{\partial m_y}{\partial y} + \frac{\partial m_{xy}}{\partial x} = v_y
\end{aligned}
\tag{2.2}
$$

2.2. Reactions

The reactions along an edge or support line, which is not parallel to either of the axes can be determined from Figure 2.3.

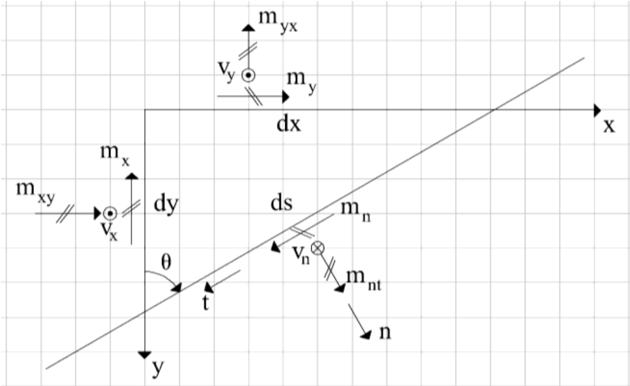


Figure 2.3. Generalised stresses along a line

The equilibriums lead to the formulas for the moment, torsion and shear force along the line

$$
\begin{aligned}
v_n &= v_x \cos\theta + v_y \sin\theta \\
m_n &= m_x \cos^2\theta + m_y \sin^2\theta + m_{xy} \sin 2\theta \\
m_{nt} &= -½(m_x - m_y)\sin 2\theta + m_{xy} \cos 2\theta
\end{aligned}
\tag{2.3}
$$

The plate may be supported along this line, in which case the reaction (r_i) along the line and the bending moment along this line (m_i) can be determined as

$$
\begin{aligned}
v_n + \frac{\partial m_{nt}}{\partial s} &= r_i \\
m_n &= m_i
\end{aligned}
\tag{2.4}
$$

The distributed reaction r_i depends on the derived of the torsion moment. If we integrate this reaction over a certain length, then we estimate the sum of the reactions over this length and we observe that there must be a single, concentrated reaction at any sharp corners as

$$R_{12} = \int_{s1}^{s2} r_i ds = \int_{s1}^{s2} (v_n + \frac{\delta m_{nt}}{\delta s}) ds = \int_{s1}^{s2} v_n ds + \left[m_{nt}\right]_{s1}^{s2} = m_{nt}^{(2)} - m_{nt}^{(1)} \tag{2.5}$$

where s_1 and s_2 are the positions on the two sides of the corner and therefore infinitely close to each other. Two torsion moments are the torsion moments at each of the two sides of the corner.

2.3. Boundary conditions

The plate boundary conditions may be free edges, simply supported, intermediate or fixed supports as shown in Figure 2.4.

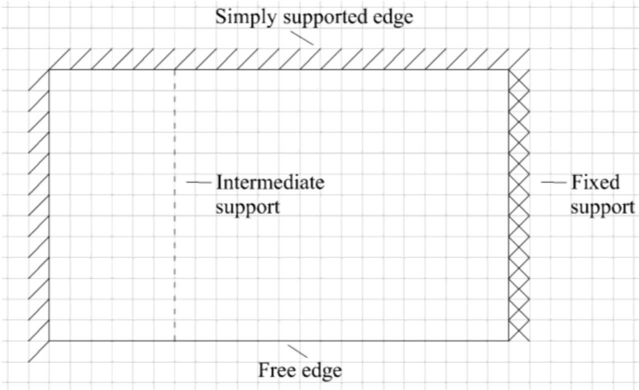


Figure 2.4. Support conditions and signatures.

Free (unsupported) edge	Simply supported edge	Intermediate support	Fixed support
Bending moment m_n=0 Torsion moment m_{nt}=0 Reaction r_i=0	Bending moment m_n=0 Deflection u=0	Deflection u=0 Bending moment $m_{n,left}=m_{n,right}$	Deflection u=0

Table 2.1. Support conditions for different support types

2.4. Resume for a rectangular plate

In the case of a rectangular plate, the equations and conditions can be expressed in Figure 2.5.

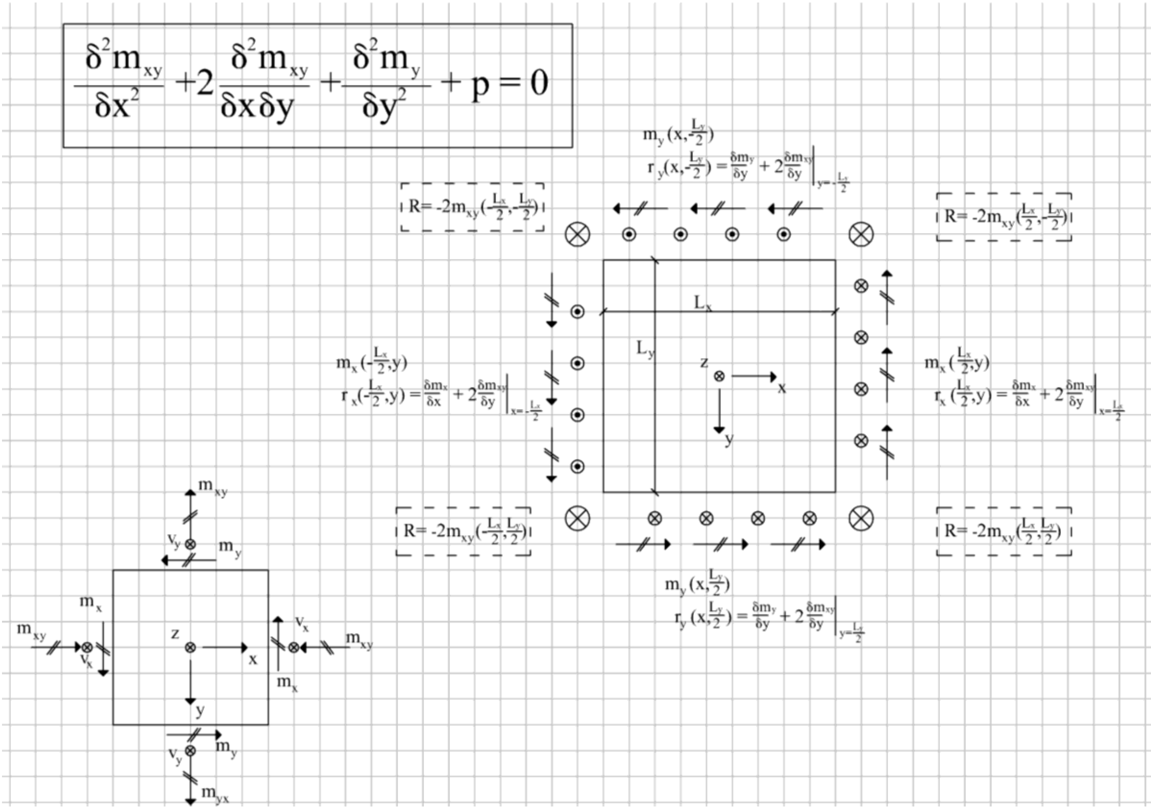


Figure 2.5. Equations and conditions for a rectangular plate.

2.5. Plate deformations

The bending and torsion moments can in a linear elastic plate be estimated from the curvatures as

$$\begin{aligned} m_x &= D(\kappa_x + \nu\kappa_y) \\ m_y &= D(\kappa_y + \nu\kappa_x) \\ m_{xy} &= D(1-\nu^2)\kappa_{xy} \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} \kappa_x &= -\frac{\partial^2 u}{\partial x^2} \\ \kappa_y &= -\frac{\partial^2 u}{\partial y^2} \\ \kappa_{xy} &= -\frac{\partial^2 u}{\partial x \partial y} \end{aligned} \tag{2.7}$$

and ν is Poissons ratio, t is the thickness and where D is the bending stiffness per width, estimated for a homogenous, uncracked plate as

$$D = \frac{Et^3}{12(1-\nu^2)} \tag{2.8}$$

The stresses in the plate are estimated as

$$\begin{aligned} \sigma_x &= \frac{12 m_x}{t^3} z \\ \sigma_y &= \frac{12 m_y}{t^3} z \\ \tau_{xy} &= \frac{12 m_{xy}}{t^3} z \end{aligned} \tag{2.9}$$

where z is the distance from the plates central plane.

This leads to a differential equation for the linear elastic plate as follows

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = \frac{p}{D} \tag{2.10}$$

2.6. How to deal with a concrete slab

The current formulas deal with the static equilibrium, which must be fulfilled by all slabs, but may be difficult to fulfil with simple models and calculations for more complex slabs. It deals also with the aspects of deformations, which may be a problem due to the non-linear relationships in steel, concrete and in the reinforced and cracked concrete cross-section.

Uncracked: At low loads, the slab will be uncracked and we may use the classic plate theory. When the tensile stresses exceed the tensile strength, the slab will start to form cracks, however, these cracks will normally only be visible and affect the deformations, when the tensile stresses exceed the tensile strength significantly.

Cracked: At higher loads, the slab will be partly cracked with an increasing number of cracks at increasing loads, leading to failure in the end. This failure will be similar to the failure in beams, provided we look at strips of the slab perpendicular to the cracks: The failure will be a combination of crushing of the concrete and yielding of the tensile reinforcement, perpendicular to the crack.

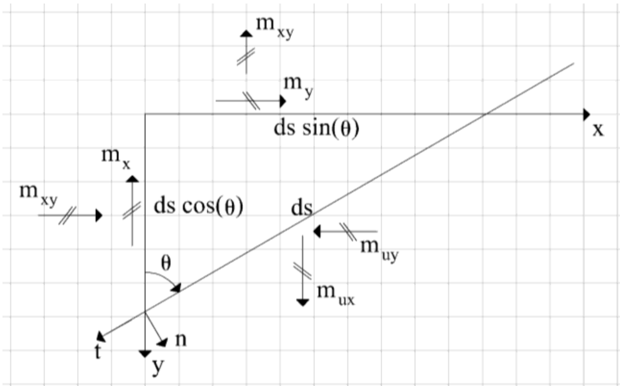


Figure 2.6. Condition at failure along the crack.

If we look at a section of the slab shown in Figure 2.6, then we can set up the equilibrium in the case of positive bending (tension in lower side of the slab)

$$m_x ds\cos\theta + m_{xy} ds\sin\theta = m_{ux} ds\cos\theta$$
$$m_y ds\sin\theta + m_{xy} ds\cos\theta = m_{uy} ds\sin\theta \qquad (2.11)$$

where m_{ux} and m_{uy} denotes the positive bending moment capacities in the x and y directions.

This leads to the failure or yield condition in a positive failure or yield line

$$-(m_{ux}-m_x)(m_{uy}-m_y)+m_{xy}^2=0 \qquad (2.12)$$

A similar expression can be found for failure with a negative bending or negative yield line, where $m_{ux}^{'}$ and $m_{uy}^{'}$ denotes the negative bending moment capacities in the x and y directions

$$-(m_{ux}^{'}+m_x)(m_{uy}^{'}+m_y)+m_{xy}^2=0 \qquad (2.13)$$

2.7. Additional reading material

2.1. Timoshenko, S.P and Woinowsky-Krieger, S.: ”Theory of plates and shells”, McGraw-Hill International Editions.
This is an old classic and well known book, which presents a number of solutions for elastic plates, which may be helpful for some of the simplest cases.

2.2. Any textbook from a plate theory course can be recommended for further knowledge and understanding of plate theory.

2.3. Any Finite Element Method program with plate elements will allow calculations of a large range of linear elastic, isotropic plates as a modern alternative to Timoshenko’s book.

3. Serviceability Limit State, deflections

One of the reasons, why we are looking into the behaviour of slabs with double spans is that we know that the double span may decrease the deflections significantly. In our simple test of the quadratic, isotropic plate, we managed to reduce the deflection with a factor of almost 3 ! – we do, however, need to be able to estimate the deflections without having to test them every time.

The last chapter presented the plate theory briefly, including the relationship between the generalised stresses (bending and torsion moments) and the generalised strains (bending and torsion curvatures) and related these to the stresses and strains in the material.

We will in this chapter determine the deflections in a very simple elastic plate and in a plate with a slightly more complex geometry. We will also have a close look at the stiffness (D or EI) used in the estimation of the deflections.

3.1. Bending stiffness of the reinforced concrete slab

We did carry out a small test with a linear elastic plexiglass plate in the classroom, where we saw the effects of the double span. We know, however, that reinforced concrete cross-sections tend to crack in the tensile zone and that this results in a quite nonlinear relationship between the bending moment and the curvature in a beam.

We should therefore start our investigations by checking the load-deflection relationship of a reinforced concrete slab, as e.g. a quadratic slab supported along all four sides and exposed to a distributed load as shown in Figure 3.1 below.

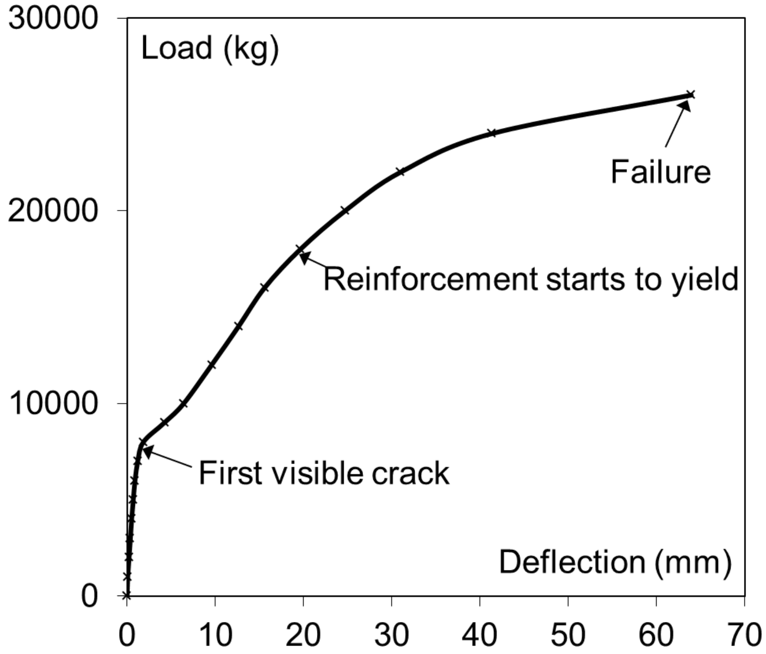


Figure 3.1. Typical variation of the deformation of a slab during testing.
A quadratic slab with simple supports [3.5, slab 822].

We can easily see that this slab has a linear relationship at the uncracked state, which reach up to 7000 kg / 26300 kg = 27 % of the failure load, where it cracks. The relationship changes in the cracked state to another linear relation, which may be described by bending in the cracked state and this continues up to 18000 kg / 26300 kg = 68 % of the failure load, where the reinforcement starts to yield.

The load levels in the Serviceability Limit State, where deflections are of importance, are normally well below the failure load and will often, but not always, be in the uncracked state.

3.2. Short term and long term loading

We need also to realise that there is a difference between the level of the loads, which only acts for a short period (minutes, hours or days) and the level of the loads, which acts for a longer time (years).

The deflections for the short term loads must be estimated with short-term modulus of elasticity, whereas the deflections for the long term loads must be estimated with the long term modulus of elasticity, so

- short term deflections are estimated for the high, short term value of the load using the concretes normal modulus of elasticity ($E_{c,short}=E_{cm}$), but

- long term deflections are estimated for the lower long term value of the load using long-term values of the modulus of elasticity ($E_{c,long}=E_{cm}/(1+\text{creep factor})$, which on the safe side can be taken as $E_{c,long}=E_{c,short}/4$)

3.3. Bending stiffness estimation

In design for practical structures, the engineer will often assume that the slabs are uncracked and use D as bending stiffness as

$$D = \frac{E_c t^3}{12(1-\nu^2)} \tag{3.1}$$

this will normally provide a fair estimate of the deflections at modest loads as seen from Figure 3.1, although the slab is usually more or less cracked even in the serviceability state. However, if the engineer wishes to be absolutely sure to obtain a conservative estimate, then it will be assumed that the cross-section is cracked, only the tensile reinforcement is taken into account and D will be replaced by EI/width, where EI is estimated for the cracked cross-section for short-term or long term loads as

$$EI = E_c I_t$$

$$I_t = \frac{1}{12}x^3 + x(x/2)^2 + \alpha A_s (d-x)^2$$

where the height of the compressionzone x is found from

$$S_t = x(-x/2) + \alpha A_s (d-x) = 0 \Rightarrow x \tag{3.2}$$

and where α is defined as

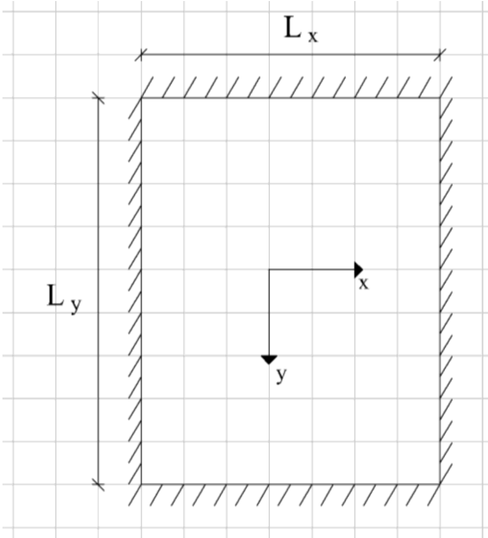
$$\alpha = \mathrm{E}_s / E_c$$

The above expressions can be derived for a beam with a cross-section in pure bending with no compressive reinforcement, where

A_s is the amount of tensile reinforcement per width b
d is the effective height of the reinforcement
E_s is the modulus of elasticity of the reinforcement
E_c is the modulus of elasticity of the concrete (short term or long term, depending of the load duration)

3.4. Rectangular slab with simple supports, solved by perturbation method

We will investigate the simple, but common, case of a rectangular slab, simply supported along all four sides and exposed to a uniform load p.



Figur 3.2. Rectangular slab with support conditions and coordinate system.

We can solve the differential equation (also called the equilibrium equation) and the boundary conditions in this case

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = \frac{p}{D}$$

$$\begin{aligned} &m_x(x = -L_x/2, y) = m_x(x = L_x/2, y) = 0 \\ &u(x = -L_x/2, y) \quad = u(x = L_x/2, y) \quad = 0 \\ &m_y(x, y = -L_y/2) = m_y(x, y = L_y/2) = 0 \\ &u(x, y = -L_y/2) \quad = u(x, y = L_y/2) \quad = 0 \end{aligned} \tag{3.3}$$

The solutions to the differential equation with p=0 are cosinus and sinus functions with different wavelengths, but we need to deal with the boundary conditions and the uniform distribution of p.

We may therefore use a perturbation method to describe the deformation u, and as we notice that the loading and the boundary conditions are double symmetric, then we will limit our solutions to the double symmetric deformations functions of u

$$u = \cos\left(\frac{m\pi x}{L_x}\right)\cos\left(\frac{n\pi y}{L_y}\right) \quad n = 1,3,5,\ldots \text{ and } m = 1,3,5,\ldots \tag{3.4}$$

which all fulfil the boundary conditions and also fulfil the differential equation for a distributed load of

$$p(x,y) = \left[\left(\frac{m\pi}{L_x}\right)^4 - 2\left(\frac{m\pi}{L_x}\right)^2\left(\frac{n\pi}{L_y}\right)^2 + \left(\frac{n\pi}{L_y}\right)^4\right]\cos\left(\frac{m\pi x}{L_x}\right)\cos\left(\frac{n\pi y}{L_y}\right) \tag{3.5}$$

We may use a combination of these solutions to derive a solution for an uniform p-load as follows

$$u = \frac{16p}{\pi^6 D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{m\pi x}{L_x}\right)\cos\left(\frac{n\pi y}{L_y}\right)}{mn\left(\frac{m^2}{L_x^2}+\frac{n^2}{L_y^2}\right)^2} \quad \text{where } m = 1,3,5,\ldots \quad n = 1,3,5,\ldots \tag{3.6}$$

This method for solving the equations and boundary conditions were much used in the older days (when I was a student in the last millenium), where computers were few and slow or did not exist and where engineering therefore had to develop this kind of solution, where a number of the contributions had to be taken in order to reach a fairly accurate result. The current solution is derived by Navier and is presented in the classic handbook [3.1], where Timoshenko and Woinowsky-Krieger have presented a number of this type of solutions.

Inserting the solution for u in the last chapters definitions of the curvatures and the shear forces and moments, we find the maximal values of these

$$
\begin{aligned}
&u_{\max} = \alpha \frac{pL_x^4}{D} && \text{at } (x,y) = (0,0) \\
&m_{x,\max} = \beta p L_x^2 && \text{at } (x,y) = (0,0) \\
&m_{y,\max} = \beta_1 p L_x^2 && \text{at } (x,y) = (0,0) \\
&v_{x,\max} = \gamma p L_x && \text{at } (x,y) = (0, \pm½ L_y) \\
&v_{y,\max} = \gamma_1 p L_x && \text{at } (x,y) = (\pm½ L_x, 0) \\
&r_{x,\max} = \delta p L_x && \text{at } (x,y) = (0, \pm½ L_y) \\
&r_{y,\max} = \delta_1 p L_x && \text{at } (x,y) = (\pm½ L_x, 0) \\
&R = npL_x^2 && \text{at all four corners}
\end{aligned}
\qquad (3.7)
$$

The estimated coefficients are listed in the Figure 3.3. below.

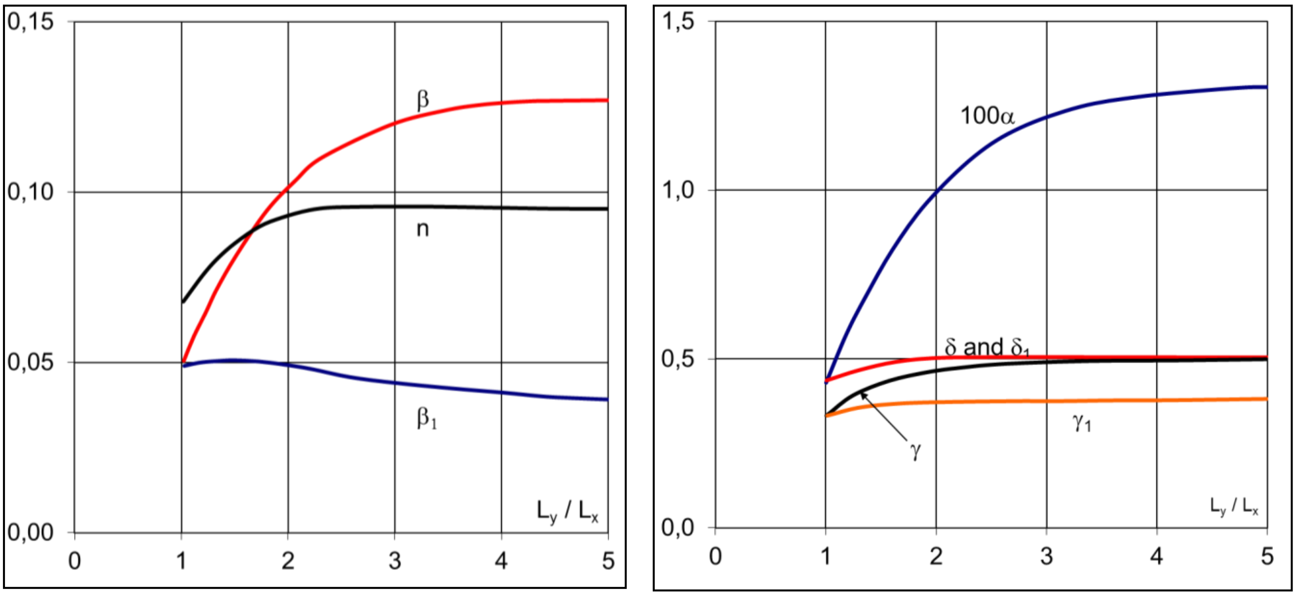


Figure 3.3. Coefficients for deflections and generalised stresses [3.2].

We may now estimate the deflections of the slab, provided that we know the slabs stiffness D for the uncracked slab or EI for the cracked slab (where EI/width replaces D in the formulas). We may also estimate the moments and thus the tensile stresses in the concrete and evaluate, whether the cross-sections cracks or not.

This is used in example 3.1.

3.5. Slabs with more complicated geometry, solved by FEM

If we deal with a more complicated load pattern or a more complicated slab as eg. a rectangular slab with three sides supported and a hole, then it becomes obvious, that we can't estimate the deflections with the old methods used in the previous example.

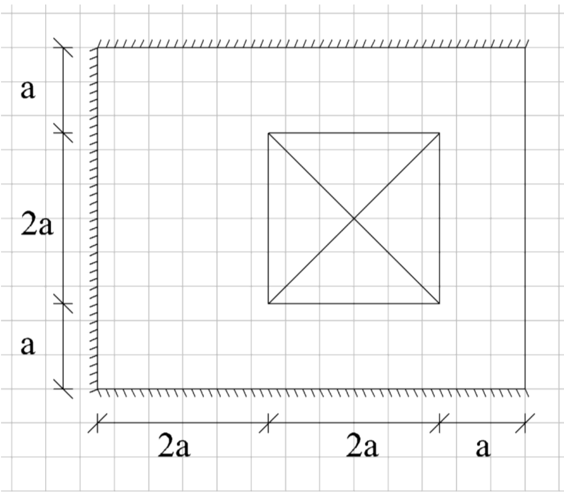


Figure 3.4. Rectangular slab with a hole, three sides supported and a hole.

Such slabs are normally analysed with a linear FEM-program, where the slab is subdivided into a number of rectangular or triangular elements, which describes the slabs geometry. A number of such programs exists and may estimate the maximal deflection.

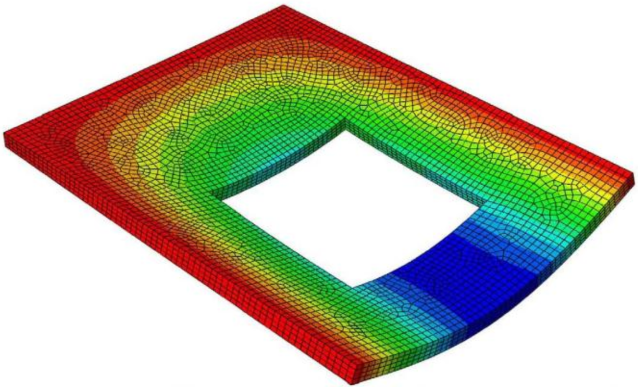


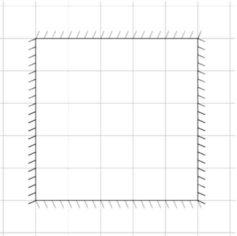
Figure 3.5. Division of the slab and variations of the deflections [3.5].

The slab on figure 3.5 was analysed with the FEM-program Abaqus for an isotropic slab with dimensions 4 by 5 m and a stiffness D. The model is linear elastic and the load and the stiffness may be replaced by other values since the maximal deflection depends on the parameters as

$$u_{\max} = factor \cdot load \,/\, stiffness \qquad (3.8)$$

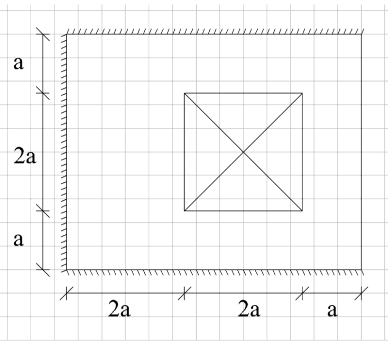
This is used in example 3.2.

3.6. Additional examples and problems



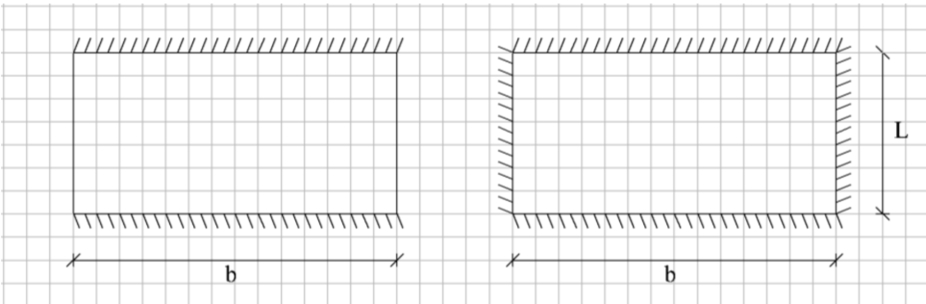
Example 3.1: Quadratic slab, loaded by a uniform load and with a given uniform reinforcement.

Recommended reading for the understanding of how to estimate deflections in the uncracked and the cracked state.



Example 3.2: Rectangular slab with a hole and supports along 3 sides, loaded by a uniform load and with a given uniform reinforcement arrangement.

Recommended reading for the understanding of how to use FEM-models for slightly more complex slabs.



Exercise B11-13
(in Danish)

3.7. Additional reading material

3.1. Timoshenko, S.P and Woinowsky-Krieger, S.: ”Theory of plates and shells”, McGraw-Hill International Editions.
This is an old classic book, which presents a number of solutions for elastic plates, which may be helpful for some simple cases.

3.2. Teknisk Ståbi, Ny Teknisk Forlag A/S, Denmark
This is an engineering handbook available in Danish and it will normally be a book, that any civil or building engineering student will have at this level of their studies and similarly, must professional engineers will have. Most other countries have similar handbooks, which often contains a few solutions for elastic plates.

3.3. Any Finite Element Method program with plate elements. Such a program can estimate the deflections of a large range of plate. This is a modern and realistic alternative to Timoshenko’s book.

<u>References</u>

3.4. Bach, C. and Graf, O.: ”Tests with simply supported, quadratic reinforced concrete plates” (In German:” Versuche mit allseitig aufliegenden, quadratischen und rechteckigen eisenbetonplatten”), Deutscher Ausshuss für Eisenbeton, Heft 30, Berlin 1915.

3.5. Mehlsen, H.: “Calculations of partly cracked concrete plates”, DTU Byg, February 2011.

4. Ultimate Limit State. Lower-limit solution with a guessed solution

The plate theory describes the differential equation and the boundary conditions, which must be fulfilled in order to fulfil the equilibrium and the constitutive conditions. This will require complex models for more complex geometries or load conditions, but it is possible to establish simple, analytical solutions for simple geometries and load distributions.

This may be achieved by a lower limit solution: A lower limit solution is a solution, which fulfils the condition of equilibrium and also fulfils the boundary conditions. This means that the solution is one of the many possible ways of transporting the loads to the supports through the plate and at the same time have equilibrium between the loads, the forces and the reactions in every point of the plate. A lower limit solution may not be the optimal solution and other solutions may verify a higher load-carrying capacity, but it will be on the safe side.

The reinforced concrete slab shall then later be checked, so it can be verified that the slab has a sufficient load-carrying capacity to carry the cross-sectional forces, predicted by the lower-limit solution.

We will therefore look at the simple and general case of a rectangular slab, simply supported along all four sides and loaded by a uniform load p.

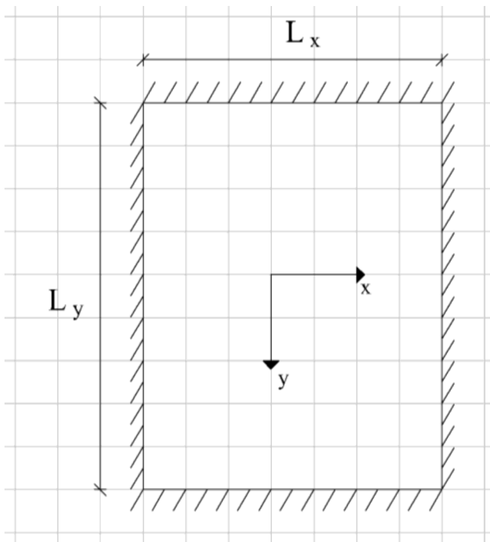


Figure 4.1. Geometry of rectangular slab.

The plate equations and conditions were listed in the last chapter for a rectangular plate as

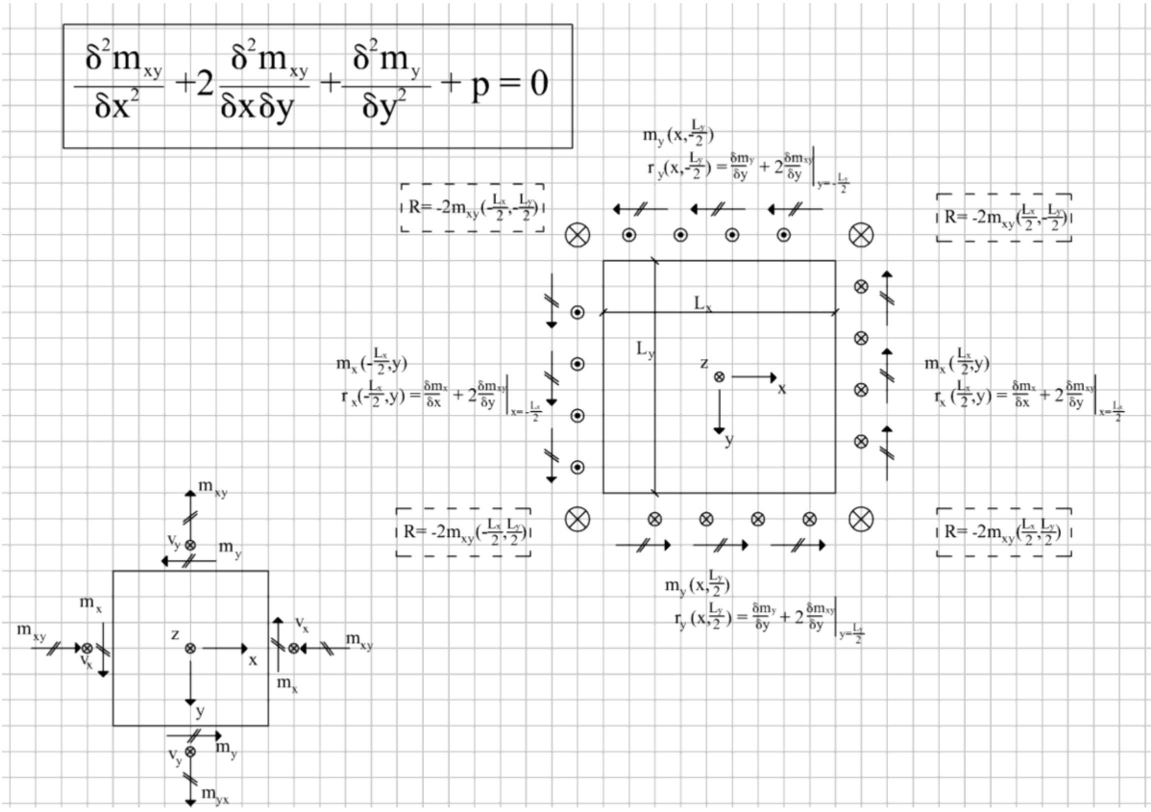


Figure 4.2. Plate equation and boundary conditions.

We will in the following develop such a lower limit solution by guessing a solution for a simple, but common case and after that we will see how solutions can be developed for more general cases.

4.1. The guessed solution for a rectangular slab with 4 sides supported

In the case of a rectangular slab with supports on all four sides, loaded by a uniform load p it would be a good guess to assume that m_x and m_y have parabolic variations (m_x=a+bx+cx^2, my=d+ey+fy^2) as this would lead to constant values of the derived in the equilibrium equation.

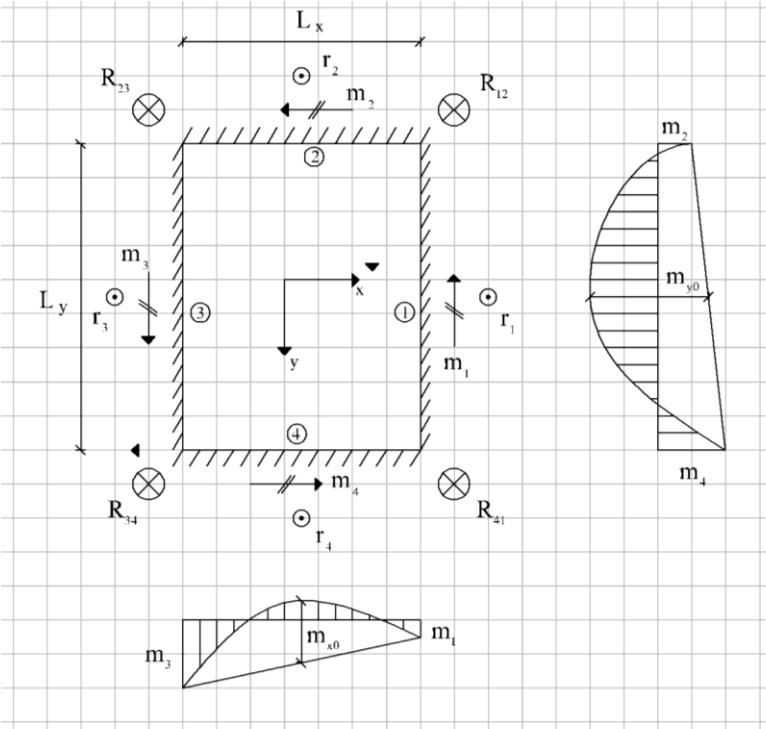


Figure 4.3. Rectangular slab with support conditions and reactions.

In the slab shown in the figure above, it would be natural to guess at variations, corresponding to the situations of a single spanning slab in the x, respectively the y-direction. This would lead to

$$\begin{aligned} m_x &= m_{xo}\left[1-4\left(\frac{x}{L_x}\right)^2\right]-\frac{m_1+m_3}{2}+(m_3-m_1)\frac{x}{L_x} \\ m_y &= m_{yo}\left[1-4\left(\frac{y}{L_y}\right)^2\right]-\frac{m_2+m_4}{2}+(m_4-m_2)\frac{y}{L_y} \end{aligned} \tag{4.1}$$

where m_{xo} and m_{yo} are constants and m_1, m_2, m_3 and m_4 are the bending moments at the (eventually) fixed supports along the four sides.

The variation of the torsional moment can be guessed as well (m_{xy}=g+hx+iy+jxy) and found to be

$$m_{xy} = m_{xyo} \frac{xy}{l_x l_y} \tag{4.2}$$

It can be shown that setting

$$m_{xyo} = -(m_{xo} + m_{yo}) \tag{4.3}$$

leads to the highest lower limit solution, that is the highest p-value for given m_{xo} and m_{yo}.

Inserting (4.1) to (4.3) in the plates differential equation (2.1) leads to

$$\left(1+4\frac{L_y}{L_x}\right)m_{xo} + \left(1+4\frac{L_x}{L_y}\right)m_{yo} = ½\, pL_xL_y \tag{4.4}$$

This solution leads to the reactions

$$\begin{aligned}
r_1 &= \frac{pL_x}{2} - 4m_{yo}\frac{L_x}{L_y^2} + \frac{m_1 - m_3}{L_x} \\
r_2 &= \frac{pL_y}{2} - 4m_{xo}\frac{L_y}{L_x^2} + \frac{m_2 - m_4}{L_y} \\
r_3 &= \frac{pL_x}{2} - 4m_{yo}\frac{L_x}{L_y^2} + \frac{m_3 - m_1}{L_x} \\
r_4 &= \frac{pL_y}{2} - 4m_{xo}\frac{L_y}{L_x^2} + \frac{m_4 - m_2}{L_y} \\
R_{12} &= R_{23} = R_{34} = R_{41} = ½(m_{xo} + m_{yo})
\end{aligned} \tag{4.5}$$

4.2. Other guessed solutions

It will of course be possible to develop other, guessed solutions for this and other problems and other load types, but it is only in few cases, that it will be possible to develop analytical lower-limit solutions.

A new and modern possibility is, however, to use a Finite Element Method (FEM) program to determine a solution, as such programs will determine solutions which fulfils the conditions (see example 3.2).

Using a FEM program will normally provide valid lower limit solutions, since the program's solutions fulfil both the equilibrium and the boundary conditions with a sufficient accuracy. The program's use of a simple stress-strain relationship will, however, of make the programs solutions extra conservative and a resulting in a higher consumption of resources (concrete and especially require more reinforcement).

4.3. Practical limitations

The bending moments (m_1 to m_4) can in principle be almost freely chosen at a fixed support, but it will normally be preferred to limit the degree of restraint i, defined as the ratio between the moment at the fixed support and the maximal moment in the span.

The value of i must be within reasonable limits if excessive cracking is to be avoided. The Eurocode 2 requires therefore that the restraining moment must be set to between 1/3 and 100 % of the restraining moment estimated by an elastic analysis. This is normally ensured by choosing i as

$$i \le \begin{cases} 0,5 \\ \dfrac{0,64\, p_{\min}}{p_{\max} - 0,64\, p_{\min}} \end{cases} \tag{4.5}$$

where p_{min} and p_{max} are the highest and the lowest loads on this plate.

Using this approach it is normally acceptable to replace the maximal moment in the slab with the moment found in the middle of the span leading to

$$\begin{aligned} & i_1 = m_1 / m_x(0) \quad i_2 = m_2 / m_y(0) \quad i_3 = m_3 / m_x(0) \quad i_4 = m_4 / m_y(0) \\ & m_x(0) = m_{xo} - ½(m_1 + m_3) \\ & m_y(0) = m_{yo} - ½(m_2 + m_4) \end{aligned} \tag{4.6}$$

4.4. Additional reading material

4.1. Nielsen, M.P. and Bach, F.: ”A class of lower bound solutions for rectangular slabs”, Bygningsstatiske Meddelelser, Dansk Selskab for Bygningsstatik, Copenhagen, Denmark, Vol 3. September 1979.
(This publication contains a number of additional, guessed solutions for other support conditions).

4.2. Any FEM program, which use elastic plate elements.

5. Ultimate Limit State. The Strip Method

In most constructions projects, it will turn out that most plates (walls, floors etc.) will have holes, non-uniform distribution of the loads or not be rectangular – in which case it will be impossible to find an analytical solution.

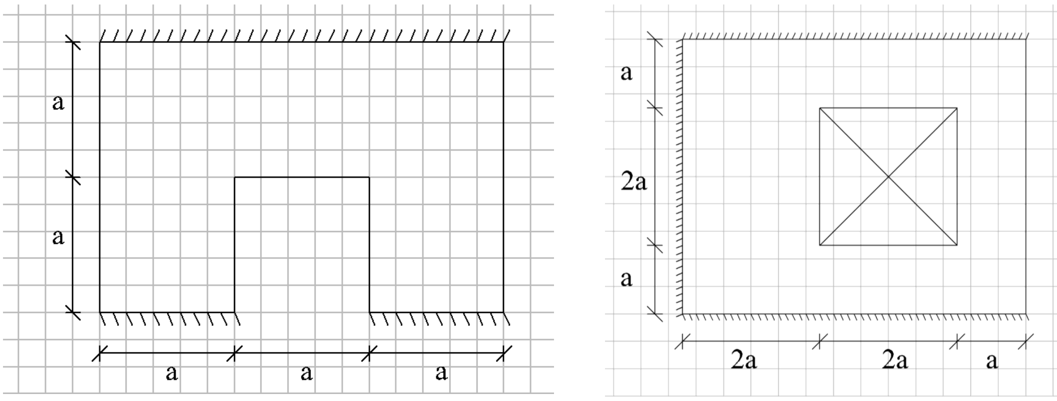


Figure 5.1. A few examples of plates with different shapes and holes.
Rectangular shape (example 5.1) and non-rectangular shape (example 5.2).

The engineer needs therefore a more general method, capable of handling the more complex cases in a safe way and this will be provided by the strip method.

5.1. The strip method

The idea behind the strip method is really to treat the plate as a system of strips (beams, plank, boards or similar), which means that each group of reinforcement bars, embedded in the concrete and parallel will be treated as an individual strip (beam).

Figure 5.2. Reinforcement arrangement in a concrete slab.

We may build up the structural model for transferring the loads through the slab to the supports by building up the plate by placing strips (beams, planks or similar) in the x and y-directions, each carrying a part of the load p as

$$\frac{\partial^2 m_x}{\partial x^2} + p_x = 0 \quad \text{and} \quad \frac{\partial^2 m_y}{\partial y^2} + p_y = 0 \Rightarrow \frac{\partial^2 m_x}{\partial x^2} + \frac{\partial^2 m_y}{\partial y^2} + p = 0, \quad \text{where} \quad p = p_x + p_y \tag{5.1}$$

where the shear forces and the reactions are determined for the strips (beams) as

$$r_x = v_x = \frac{\partial m_x}{\partial x} \quad \text{and} \quad r_y = v_y = \frac{\partial m_y}{\partial y} \tag{5.2}$$

This leads to a model of strips in different directions, each having bending and shear forces but no torsion (m_{xy}=m_{nt}=0) and provided a valid lower-limit solution as the equilibrium between the loads, the internal forces and the reactions is fulfilled in all parts of the plate. This is described in details in the examples 5.1 and 5.2.

In each strip the maximal and minimal moments ($m_{i,min}$ and $m_{i,max}$) are determined at the lower limit of the loadcarrying capacity is determined as the maximal value of $p=p_i^{(-)}$, where the moments stay within

$$-m_{iu}^{'} \leq m_{i,\min} \leq m_{i,\max} \leq m_{iu} \quad (5.3)$$

where m'_{iu} and m_{iu} are the bending moment capacity of strip i in negative bending (tension in the top of the plate) and in positive bending (tension in the bottom of the plate).

The lower limit solution for the capacity of the plate $p^{(-)}$ is determined as the lowest of the capacities for the strips

$$p^{(-)} = \min(p_1^{(-)}, p_2^{(-)}, ...) \quad (5.4)$$

This will be a lower limit solution, as the load distribution between the strips and the division of the plate into strips may be less than optimal, however, it will secure the load-carrying capacity of the plate. The strip model will ignore the plates ability to transfer torsion moments, which may further decrease the estimated load-carrying capacity.

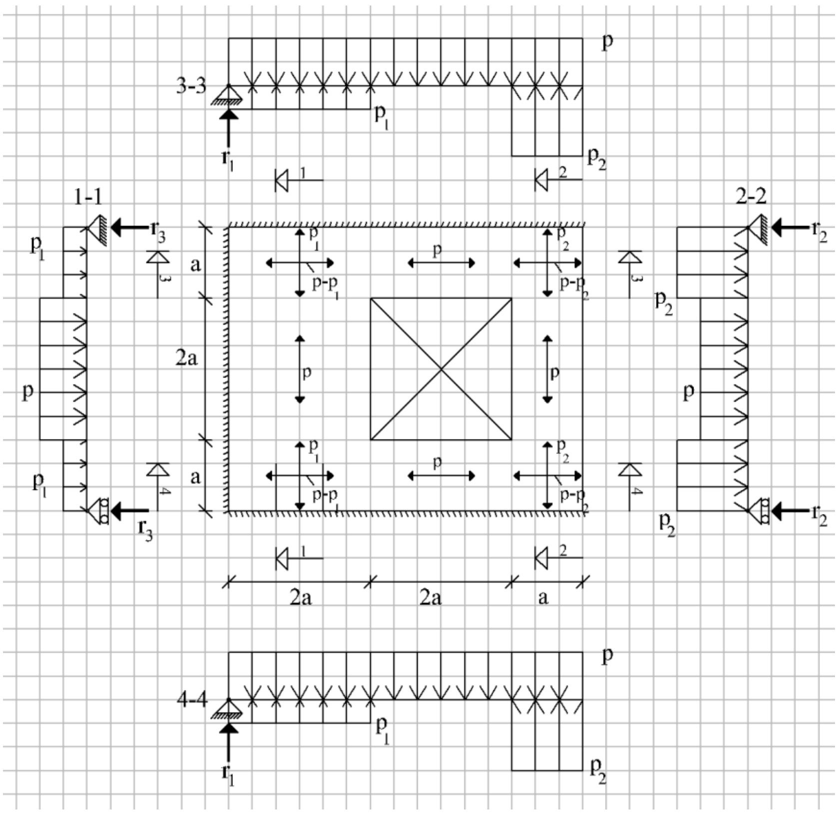


Figure 5.3. Typical model (from example 5.2) transfer of the loads and models for the strips.

A detailed description of how a model for a statically determined model is established is developed in example 5.1, whereas the more complicated, statically undetermined model above is developed in example 5.2.

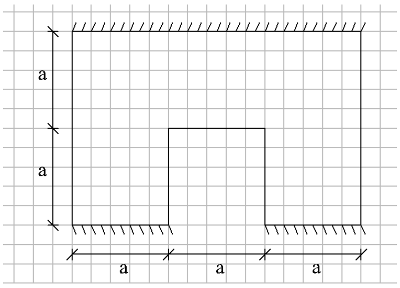
5.2. Comparison to the classic plate theory

The strip method could of course also be derived from the classic plate theory by ignoring the plates capacity for torsion, assuming m_{xy}=m_{nt}=0, which leads to

$$
\begin{aligned}
&\frac{\partial^2 m_x}{\partial x^2} + \cancel{2\frac{\partial^2 m_{xy}}{\partial x \partial y}} + \frac{\partial^2 m_y}{\partial y^2} + p = 0 \\
&\frac{\partial m_x}{\partial x} + \cancel{\frac{\partial m_{xy}}{\partial y}} = v_x \\
&\frac{\partial m_y}{\partial y} + \cancel{\frac{\partial m_{xy}}{\partial x}} = v_y \\
&v_n + \cancel{\frac{\partial m_{nt}}{\partial s}} = r_i \\
&m_n = m_i \\
&R = \cancel{m_{nt}^{(2)}} - \cancel{m_{nt}^{(1)}} \text{ (which results on no concentrated reactions at the corners)}
\end{aligned}
\tag{5.5}
$$

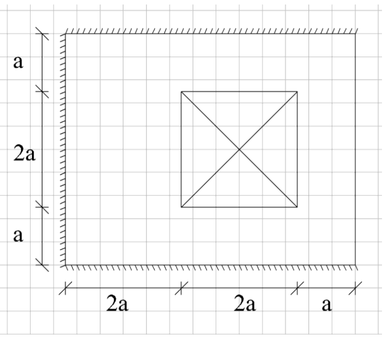
We can see that the equilibrium condition and the reactions from the classic plate theory corresponds precisely to the strip methods conditions, as long as the plates torsion capacity is ignored – or to put it in another manner, as long as the chosen model fulfils the plates equations without torsion.

5.3. Additional examples and problems



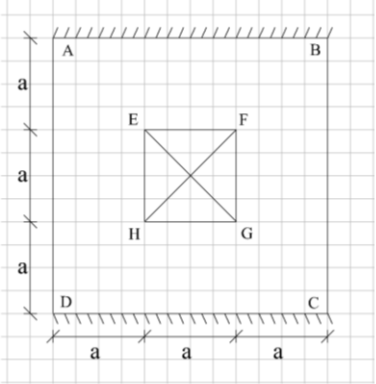
Example 5.1: C-shaped plate, loaded by a uniform load and with a given uniform reinforcement.

Recommended reading for the understanding of how to build up the model and how to carry out the calculations.

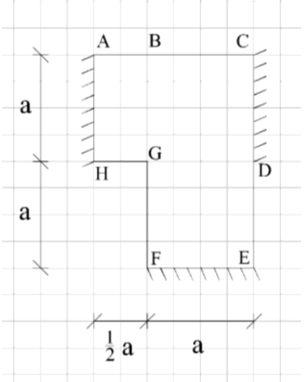


Example 5.2: Rectangular plate with a hole and supports along 3 sides, loaded by a uniform load and with a given uniform reinforcement arrangement.

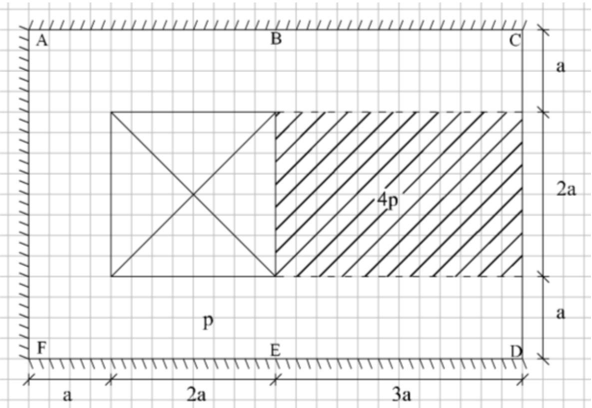
Recommended reading for the understanding of how to build up the model and how to carry out the calculations



Exercise B11-14
(in Danish)



Exercise B11-15
(in Danish)



Exercise B11-16
(in Danish)

5.4. Additional reading material

5.1. Hillerborg, A. ”Strip method design handbook”, E&FN Spon, 1996.
(Professor Hillerborg was one of the late pioneers behind the strip method and this books contain substantial amounts of explanations and illustrations of the method as well as a number of examples of more complicated or advanced use of the method)

6. Ultimate Limit State. The Yield Line Method

The load-carrying capacities of reinforced concrete slabs may be verified by many lower-limit solutions, as e.g. the guessed solutions, the Finite Element Method estimations, the strip method or many others.

The experience is, however, that the lower limit methods, suitable for simple estimations by hand tend to involve quite a lot of calculation for slabs with even just slightly complex geometries or load distributions.

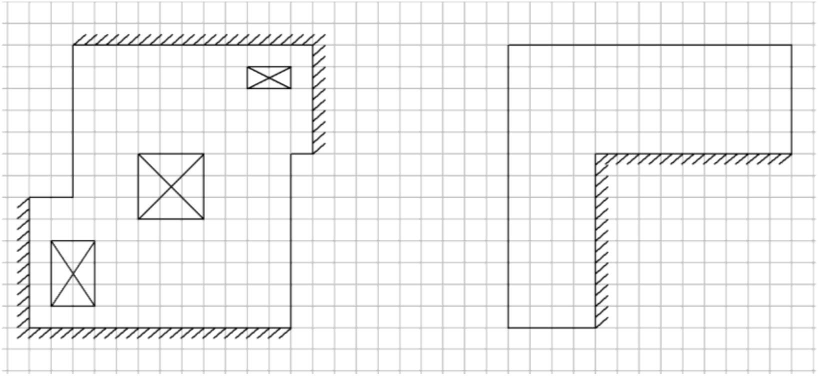


Figure 6.1. A few slabs, which would be difficult to or impossible to analyse with the strip method or with a guessed solution.

We do therefore need a simple method for handling these slabs. Such a method can be established by investigating different failure mechanisms and determine the corresponding failure loads.

6.1. The virtual work principle and the upper limit method

The upper limit method is based on an assumed failure mechanism, which is evaluated using the Virtual Work Principle. This means in simple terms, that the student, designer, engineer or computer goes through the following steps

1. chooses a possible failure mechanism,
2. estimate the external work W_e (load times deformation) of the failure mechanism and
3. estimate the internal work W_i (stress times strain) of the failure mechanism.
4. estimate the external load at which the failure mechanism is possible by setting W_e=W_i.

We will first use this method on a simple beam in order to understand the basics in the method and then move on to the more complex problem of the slabs.

6.2. Upper limit method for a simple beam

Testing of the beam and the failure mechanism

The beam has been tested in the lab and the failure mechanism has been observed as shown below

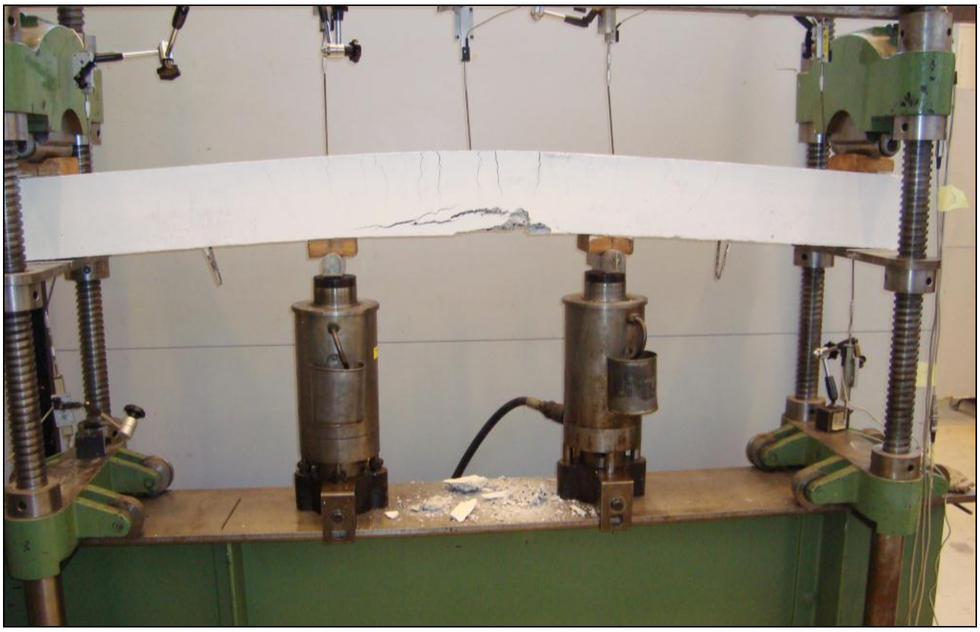


Figure 6.2. Beam with 2-point loading [6.9].

The beam is turned up-side down for practical purposes in the testing in the laboratory and has therefore the tensile zone in the top.

The beam shows yielding and significant crack formation in the marked area between the two point loads, as this part has the same, constant bending moment over this length (this can be seen in the video).

Static model and failure mechanism

The load-carrying capacity of a beam is normally estimated with simple formulas as explained in any basic course in concrete structures and should not require use of the virtual work method, but is still very suitable as an introduction.

We will therefore look at the simply supported beam shown below:

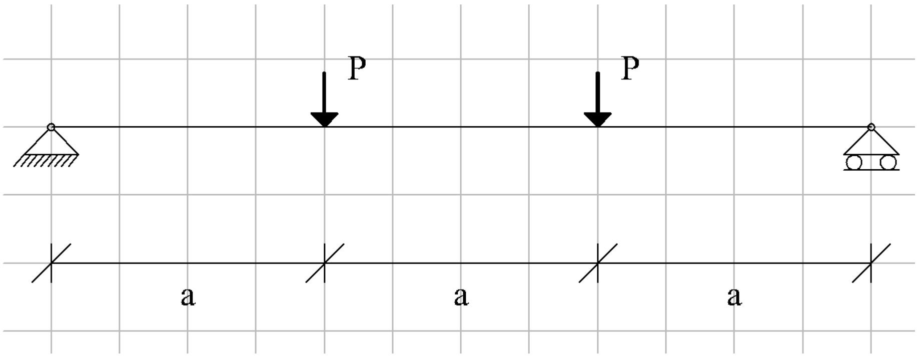


Figure 6.3. Static model of the tested beam

The test showed that the yielding may be concentrated in a minor length of the beam or it may be distributed over a length of the beam, but the actual failure would be concentrated in a small zone in the yielded area (a so called plastic hinge). This corresponds to the failure mechanism shown below

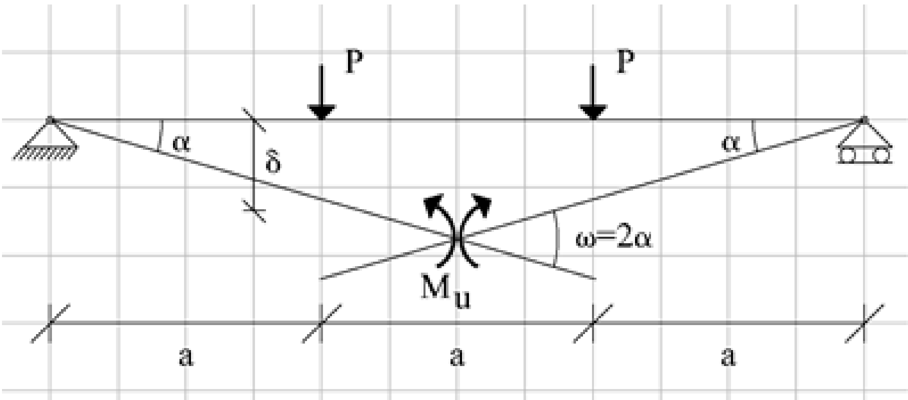


Figure 6.4. Failure mechanism in the tested beam.

We may now start using the virtual works principles and estimate outer and inner work and also estimate the load carrying capacity related to this failure mechanism.

6.2.1. The external work

The external work is easily estimated as the sum of the external loads multiplied with the incremental deformations of the beam in the failure in the load positions

$$W_e = \int p(s)\cdot\delta(s)\cdot ds + \sum P_i\delta_i \tag{6.1}$$

where

P_i is a concentrated load in point i
p(s) is a distributed line load at coordinate s
δ_i is the incremental displacement of the mechanism in position i
δ is the incremental displacement of the mechanism at coordinate s

6.2.2. The internal work

The internal work is the sum of the bending moments and the incremental curvatures κ_δ in the beam, which occurs at the instant, where the failure occurs. This means that

$$W_i = \int M(s)\cdot\kappa_\delta(s)\cdot ds \tag{6.2}$$

The part of the beam failed (reinforcement yielded and the concrete was crushed) in more or less concentrated area in the zone between the two loads, which means that the extra, incremental curvature κ_δ occurs in this zone only and is zero in all other parts of the beam. This means that the inner work in the beam with a positive yielding moment is calculated as the plastic work in the zone

$$W_i = \int M(s)\cdot\kappa_\delta(s)\cdot ds = \int M_u\cdot\kappa_\delta(s)\cdot ds = M_u\cdot\kappa_\delta\cdot a = M_u\cdot\omega \tag{6.3}$$

where

$$\int \kappa_\delta(s)\cdot ds = \omega \tag{6.4}$$

and similarly in a beam with a negative yielding moment

$$W_i = -M_u^{'}\cdot\omega \tag{6.5}$$

where

M_u is the positive yielding moment
M_u' is the negative yielding moment
ω is the bend in the plastic hinge of the beam.

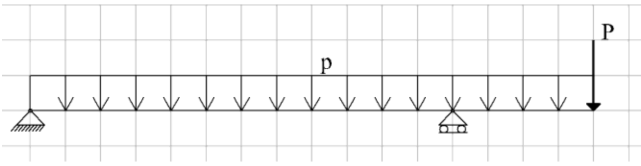
6.2.3. The loadcarrying capacity

This is found by setting the internal work W_i equal to the external work W_e, leading to an upper limit value of load-carrying capacity.

This is described in details in example 6.1.

6.3. Additional examples and problems

Example 6.1: Two-point loading of a simply supported beam tested in the laboratory. **Recommended** reading for the understanding of the upper limit solution for beams.



Example 6.2: Cantilever beam with distributed and concentrated loads, which will show how to use the method on slightly more complex problems.

6.4. Additional reading material

Virtual work or virtual displacements are normally a part of the basic building mechanics courses and further explanations of the method can be found in standard textbooks as e.g.

6.1. Hartsuijker, C. and Welleman, J.W.: "Engineering mechanics, volume 1, Equilibrium", Springer 2006. This is an introduction to building mechanics and provides a brief introduction to the virtual works principle.

6.2. Krenk, S.: "Mechanics and analysis of beams, columns and cables". This provides a slightly more detailed and focused introduction to beams and to the virtual work principle.

6.5. Upper limit solution – yield line method – in slabs

The upper limit solution in a slab is derived using the same approach as for a beam:

1. we identify and select a kinematically permissible failure mechanism (yield line pattern),
2. we calculate the external and internal work during the failure and
3. we calculate the upper limit load from the requirement of $W_e = W_i$.

6.5.1. Failure mechanisms during testing

We will look at the behaviour of a slab during testing and see how it develops cracks and in the end how it fails. We will for this purpose look at the failure mechanisms of two simple slabs:

Slab 1: A simple rectangular slab of fibre reinforced concrete tested at DTU
Slab 2: A simple quadratic slab of reinforced concrete, tested in Stuttgart

6.5.2. Testing of slab 1

The slab was placed in a steel frame, providing simple supports along all four sides and loaded uniformly by an airbag.

Figure 6.5. Rectangular slab 1 with observed shape of failure mechanism [6.7].

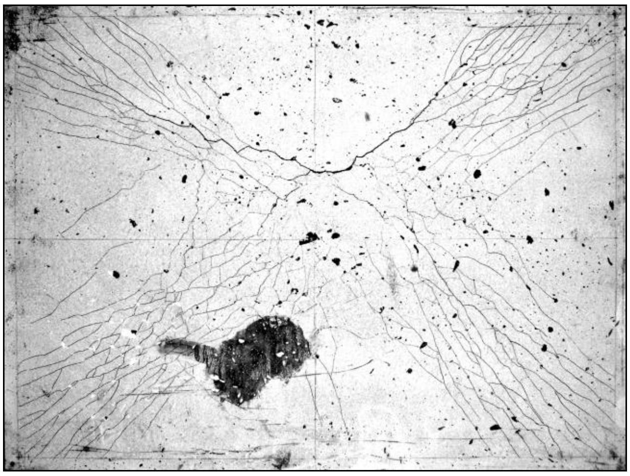


Figure 6.6 Slab 1 with observed crack formation [6.7].

The test setup did not allow a photographic registration of the crack development with increasing load, but the Figures 6.5 and 6.6 show clearly the developed cracks and the failure mechanism.

6.5.3. Testing of slab 2

The quadratic reinforced concrete slab of 2 x 2 m was supported on a simple steel frame and loaded it 16 identical point loads (as a simulation of a uniform load). The deformations and the crack patterns were registered for each load steps shown on Figure 6.9.

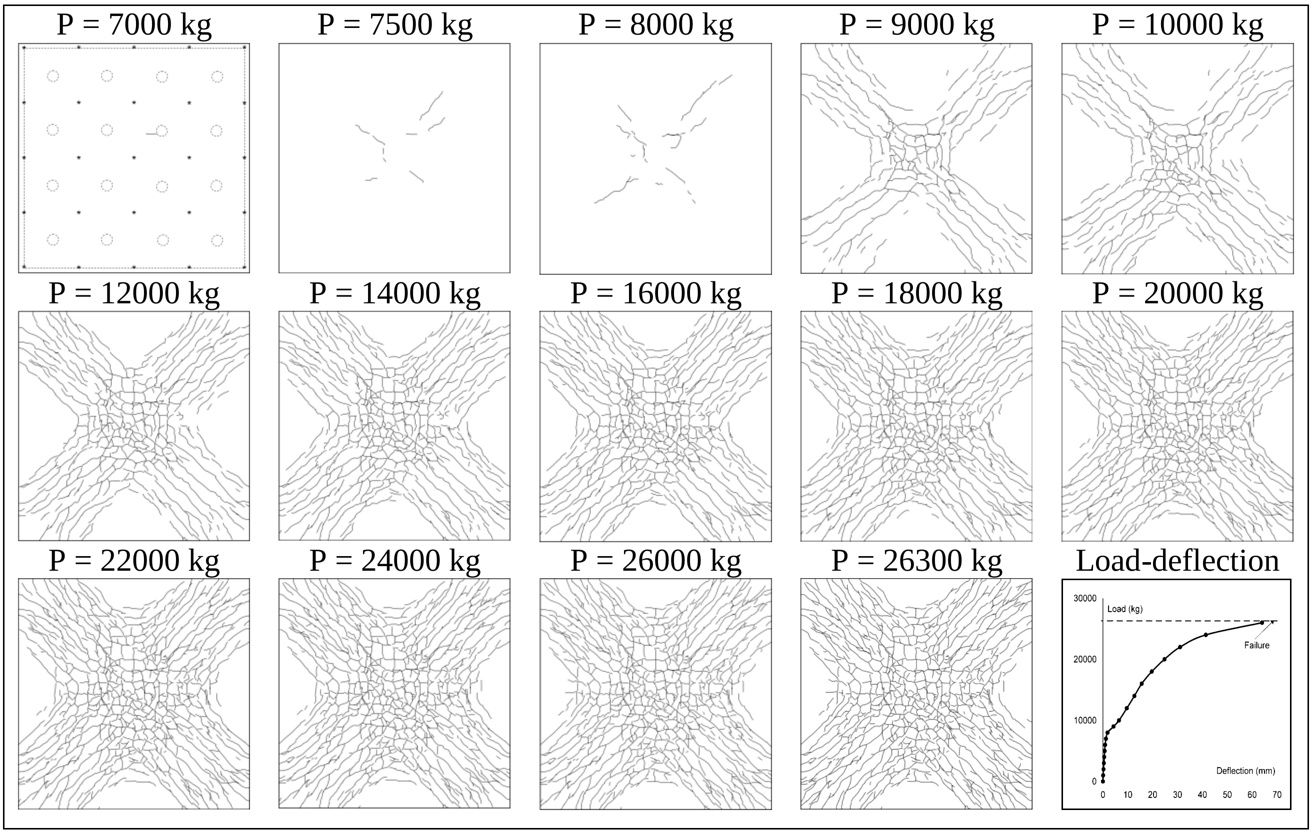


Figure 6.7. Development of cracks in concrete slab at increased load level [6.5].

6.5.4. Failure mechanisms observed in the tests

We can see that the slabs do not yield in points (hinges), but rather in lines, which allows the failure mechanism to develop.

The mechanism for the slab 2 in Figure 6.7 involves tilting of the corners as the supports allowed the corners to tilt and influenced the crack formation near the corners to some extent, whereas the mechanism for the slab 1 in Figure 6.5 and 6.6 does not include tilting of the corners as the displacements of these were prevented.

The measured deflections and registrations have identified the failure mechanisms as shown below

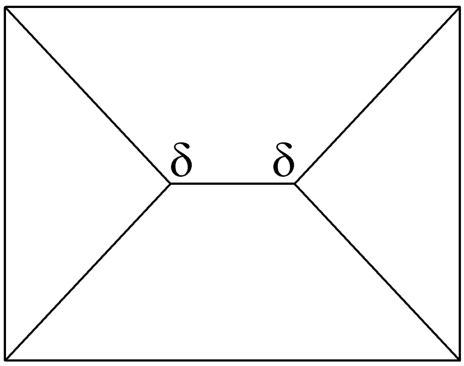


Figure 6.8. Failure mechanism in slab 1 [6.7]
Example 6.3

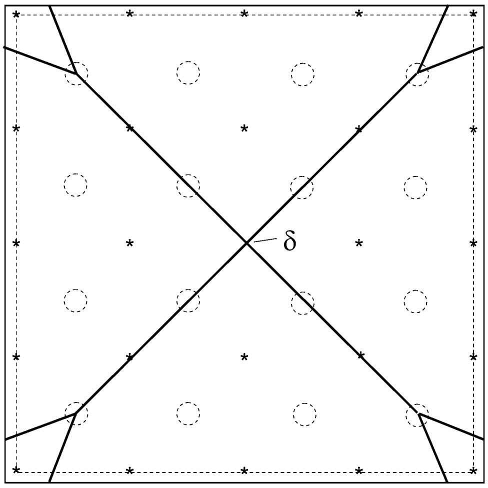


Figure 6.9. Failure mechanism in slab 2 [6.5]
Example 6.4

We can see from these tests (and many more [6.8]) that the failure mechanism in concrete slabs involves yielding in lines, with or without tilting of the corners. We will therefore use the so-called "Yield line method" [6.6] developed at DTU, where the engineer estimates the load carrying capacity for the critical failure mechanism. This leads normally to simple estimations in cases, where the lower-limit solutions require extensive calculations or are even impossible to use.

It needs to be said, that any upper limit solution predicts a load carrying capacity, equal to or above the correct failure load. This means that the failure load is correctly estimated if the engineer has identified the correct failure mechanism, slightly higher for a slightly incorrect failure mechanism, but it means also that the predicted failure load may be a lot too high if a wrong (but possible) failure mechanism is used.

The method must therefore be used with some care in the design of structures, but has been used for decades in Denmark and other countries in the design.
The advice to you is therefore to investigate several different failure mechanisms and perhaps even try to optimize those mechanisms as this will approach lead to a fairly safe and correct estimate of the load-carrying capacity. (That is actually what the old and experienced engineers do !).

6.6. Possible failure mechanisms for our models

We have seen from the tested slab, that it is very easy to determine the failure mechanism after a test have been carried out – but we would also like to be able to predict the failure mechanism without actually testing a slab in order to be able to predict the load-carrying capacity. This is to a large extent dependent of engineering judgement – often combined with the analysis of several different failure mechanisms.

We noticed that the displacements of the beam and the slab during the failure essentially consisted of

1. sections, which moved and rotated without any additional curvature and
2. plastic hinges or straight yield lines, which did curve and bend.

This leads to a few simple rules to verify if a failure mechanism is possible as illustrated in Figure 6.10, as section 1 must rotate around the axis A-B, where the displacement of any point in section 1 is proportional to the distance to the axis of rotation.

The section 2 must rotate around axis C-D and the displacement of any point in section 2 must be proportional to the distance to the axis of rotation. This means that the extrapolation of the yield line must meet the extrapolation of the two axis of rotation in a point O.

The only exception to this rule is when the two axis of rotation are parallel, in which case the yield line between the two sections must be parallel to the two axis. The two axis and the yield lines extrapolations will thus never cross each other.

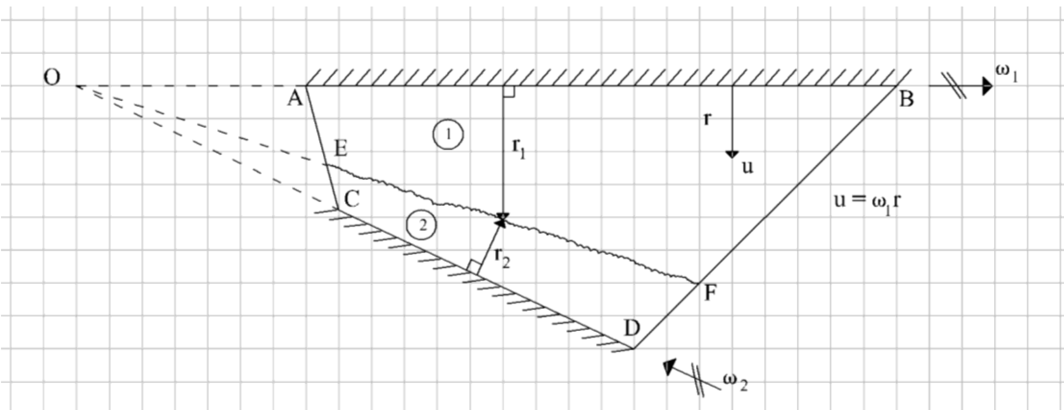


Figure 6.10. Kinematically possible failure mechanism.

The failure of a slab can result in either a part of the slab collapsing (local failure, local yield line pattern) or in all collapse of the whole slab (global failure, global yield line pattern).

You may later use example 6.5 or download the self-quiz, to train your ability to distinguish between possible and impossible yield line patterns.

6.7. The external work

This is estimated (as for the beam) as the sum of the loads multiplied with the incremental deformations of the beam in the points, where the loads are applied

$$W_e = \int p(x,y)\cdot\delta(x,y)\cdot dx\cdot dy + \sum P_i\delta_i \qquad (6.6)$$

6.8. The internal work

The work is carried out in the slab in the areas with yielding, concentrated in the yield lines, just as it was created in the plastic hinge in the beam. The work is proportional to the yielding moment and the “hinge” in the yield line as shown in Figure 6.11.

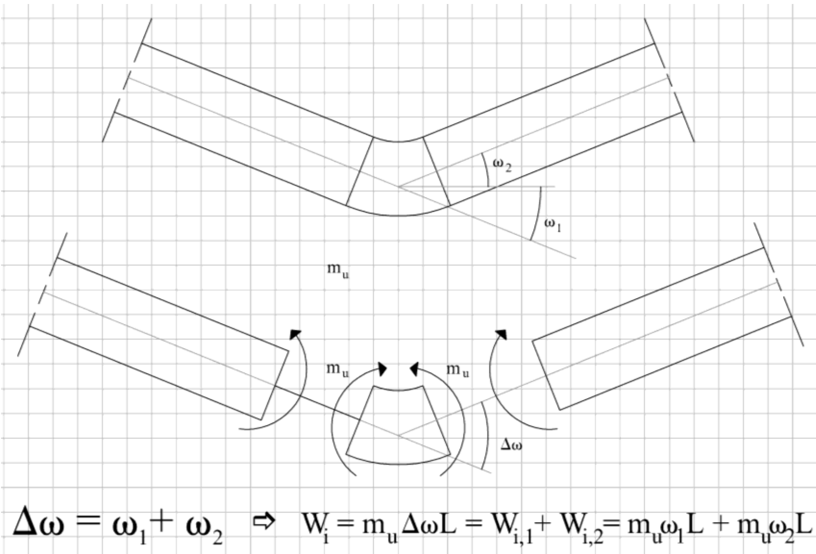


Figure 6.11. Internal work in the slabs yield line, divided into sections 1 and 2.

The area with yielding may be distributed over some width of the slab or be very local, but will in the abstract model be concentrated in a yield line – just as the yielding in the beam was concentrated in a single plastic hinge.

The internal work is the work in the yield lines (similar to the work in the plastic hinge in the beam) and calculated as

$$W_i = \int m_u\cdot\Delta\omega\cdot ds = \sum m_u\cdot\Delta\omega\cdot L \qquad (6.7)$$

where

m_u is the yielding moment per length for bending perpendicular to the yield line
$\Delta\omega$ is the bend in the yielding line
L is the length of the yield line

The problem with the estimation of the internal work in the yield lines is normally that there is a contribution from each of the yield lines and that the reinforcement directions may not be parallel or perpendicular to the yield line, so the estimation of the internal works in the yield lines may be quite extensive.

We would therefore like to simplify the estimations by separating the internal work in a yield line into the contributions from each side of the yield line. This is estimated as the work carried out by the individual sections of the slab between the yield lines as

$$W_{in} = \int m_u \cdot \omega_{in} \cdot ds = m_u \cdot \omega_{in} \cdot L$$
$$W_i = \sum W_{in} = \sum m_u \cdot \omega_{in} \cdot L \qquad (6.8)$$

6.9. The load-carrying capacity

This is now estimated (as for the beam) from

$$W_o = W_i \qquad (6.9)$$

6.10.Rotations of a section and internal work

A yield line does not have to be parallel to a convenient axis of rotation, just as the slab strength may be different in different directions – we may after all have different amounts of reinforcement in the different directions. We will therefore often wish to separate the rotation of a section in two contributions around the x and y axis as shown in Figure 6.12.

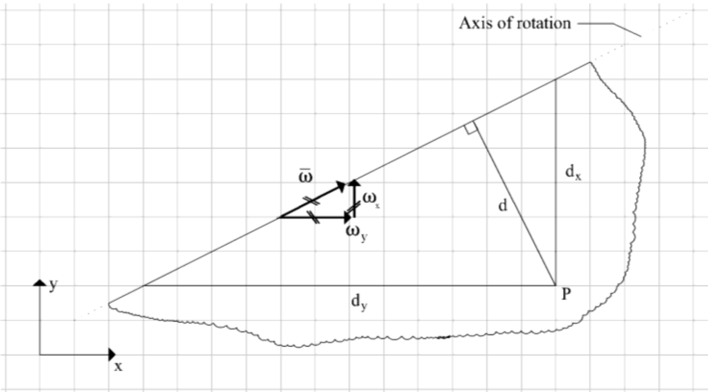


Figure 6.12. Rotations of a section.

The rotations on Figure 6.12 are defined as

$$\omega = u_p / d \quad \omega_x = u_p / d_x \quad \omega_y = u_p / d_y \tag{6.10}$$

and leads to the internal work of

$$W_i = (m_{ux}\omega_y + m_{uy}\omega_x) \cdot L \tag{6.11}$$

6.11.Broken yield lines and internal work

We can actually integrated this work from a number of the sides of a section of the plate as shown below where we find that the work for a “broken” positive (or broken negative) yield line can be estimated as the equivalent work along a straight line, going from one end to the other end of the broken yield line.

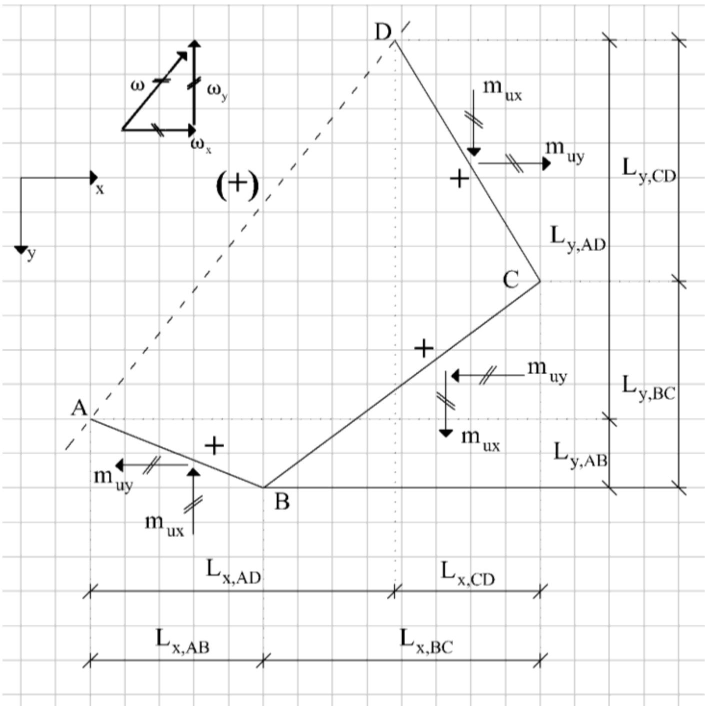


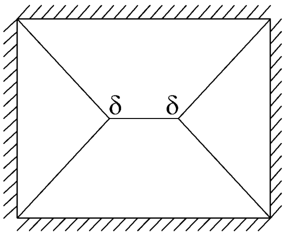
Figure 6.13. “Broken” yield line consisting of several straight yield lines, all either 100 % positive or 100 % negative lines. Lines are shown positive on figure.

The calculation of the total, equivalent internal work from the yield line through the points A-B-C-D is found as

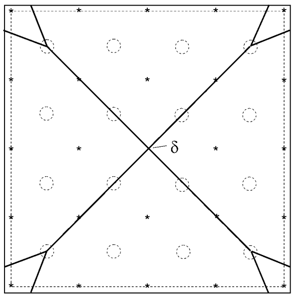
$$\begin{aligned} W_{i,ABCD} &= W_{i,AB} + W_{i,BC} + W_{i,CD} \\ &= (m_{uy}\omega_x L_{x,AB} - m_{ux}\omega_y L_{y,AB}) + (m_{uy}\omega_x L_{x,BC} + m_{ux}\omega_y L_{y,BC}) + (m_{uy}\omega_x L_{x,CD} + m_{ux}\omega_y L_{y,CD}) \\ &= m_{uy}\omega_x L_{x,AD} + m_{ux}\omega_y L_{y,AD} \end{aligned} \qquad (6.12)$$

This expression is actually quite logical, when we remember that m_{ux} and m_{uy} denote the bending moment capacities per length correspond to the bending strengths from reinforcement bars placed in the x and y directions. The equivalent internal work is thus proportional to the bend and to the number of reinforcement bars crossing the yield lines.

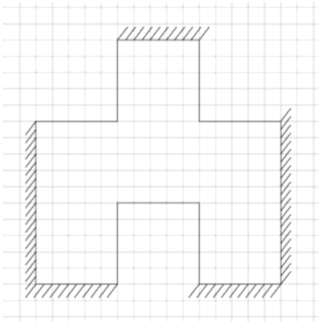
6.12. Additional examples and problems



Example 6.3: Rectangular slab tested at DTU.
Recommended reading for understanding failure mechanisms and estimations for the yield line method.

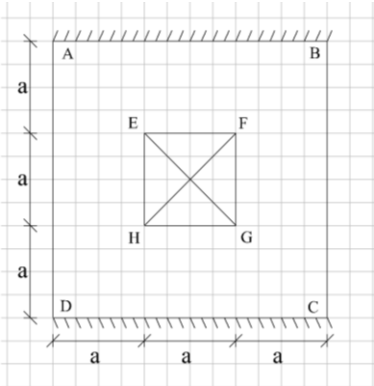


Example 6.4: Square slab, used by K.W. Johansen as a part of the documentation of the yield line method.
Recommended reading for the understanding of failure mechanisms and the effects of tilting corners and how to avoid tilting of the corners.

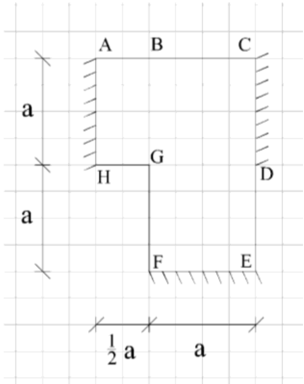


Example 6.5: Slab used as a problem at the examination, where the students should indicate a possible failure mechanism. A large number of yield line figures are presented for this plate, some are possible and some are not. Your task is to distinguish the possible mechanisms from the impossible mechanisms.

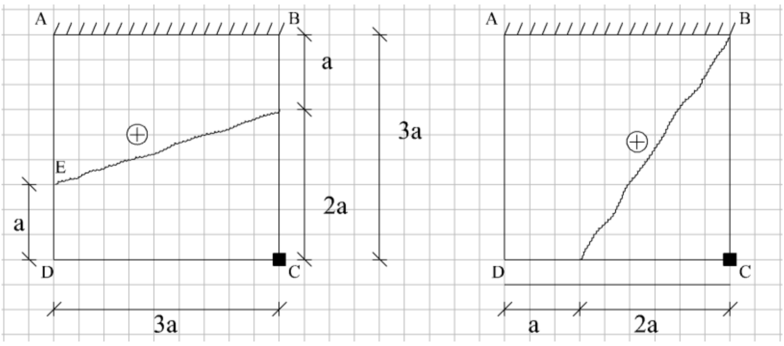
An alternative to this is to download the self-quiz, which randomly select a number of yield line patterns from a pool of over 150 figures – you can use the quiz a number of times.



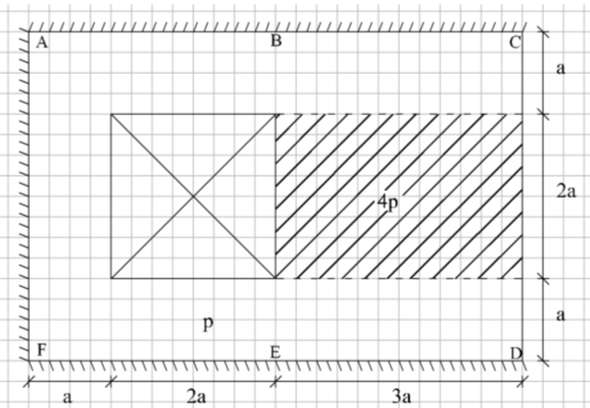
Exercise B11-14
(in Danish)



Exercise B11-15
(in Danish)



Exercise B11-16
(in Danish)



Exercise B11-17
(in Danish)

6.13.Additional reading materials

6.3.Johansen, K. W.: “Pladeformler”, Polyteknisk Forlag, Copenhagen, 1968, 14 pages (“Yield line formulae for slabs”, Translated by Cement and Concrete Association, London, 1972. Ref 12.044).
This publication contains a large number of examples, where the upper limit solution is presented, but where the actual deriving of the formulas is not presented.

6.4.Kennedy, Gerard and Goodshild, Charles: ”Practical yield line design”, British Cement Association, 2003, http://www.concretecentre.com/PDF/PYLD240603a.pdf, 171 pages.
This is a good and extensive introduction to a practical use of the yield line method.

References

6.5.Bach, C. and Graf, O.: “Versuche allseitig aufliegenden, quadratischen und rechteckigen eisenbetonplatten”, Deutscher Ausschuss für Eisenbeton, heft 30, 1915.

6.6.Johansen, K. W.: ”Brudlinieteorier”, Jul. Gjellerups Forlag, Copenhagen, 1943, 191 pages. (Yield Line theory, Translated by Cement and Concrete Association, London, 1962, 182 pages).

6.7.Knudsen, K. I.: “Slabs in fibrereinforced concrete, Use of yield line method (In Danish: Plader i fiberarmeret beton. Brug af brudlinieteori)”, Danmarks Teknisk Universitet, 2008.

6.8.Park, R. and Gamble, W. L. :”Reinforced concrete slabs”, 2ed, Wiley, 2000.

6.9.Khailani, Z.Y.; Pey, S. And Thyssen. A.A. :”Laboratoriepraktik 11761, Betonbjælke, testning og videofremstilling”, Byg-DTU, December 2008, Lyngby, Denmark.

7. Dictionary (English – Danish)

Kinematically permissible failure mechanism: Kinematisk tilladelig brudfigur
Degree of restraint: Indspændingsgrad
External work: Ydre arbejde
Failure mechanism: Brudmekanisme
Global failure : Global brudfigur, total brudfigur, total kollaps
Internal work: Indre arbejde
Local failure: Lokal brud, lokal brudfigur
Lower limit solution: Nedreværdiløsning
Plastic hinge: Flydeled
Restraining moment: Indspændingsmoment
Static conditions of equilibrium: Statiske ligevægtsbetingelser
Static equilibrium : Statisk ligevægt
Statically permissible solution: Statisk tilladelig løsning
Strip method: Strimmelmetoden
Third point positions: Trediedelspunkterne
Tilts: Vippere
Virtual works principle: Virtuelt arbejdes princip
Yield line: Flydelinie

Additional terms and words will be added, whenever a fair need is identified – and I assume, that my students will help me with this point.

8. Examples

The following examples are integrated in this document to facilitate printing and to make sure you have all the relevant information:

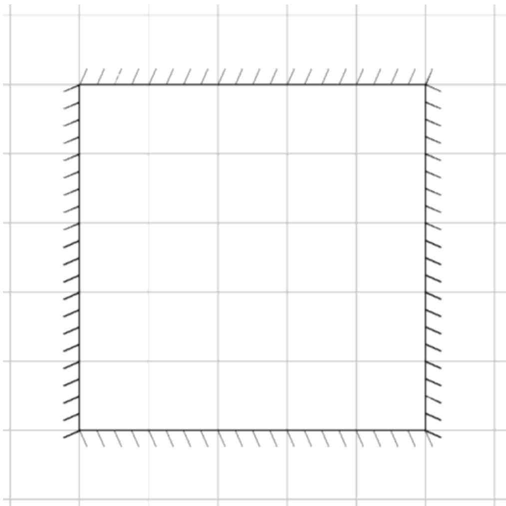
- Example 3.1. Serviceability Limit State. Deflections. Rectangular plate with simple supports and uniform loading.
- Example 3.2. Serviceability Limit State. Deflections. Rectangular plate with hole and three sides supported.
- Example 5.1. C-shaped plate with some simply supported sides and some free sides.
- Example 5.2. Strip Method Design. Rectangular plate with a hole and three sides supported.
- Example 6.1. Yield Line Method. Two point loading of a beam – plastic hinge.
- Example 6.2. Yield Line Method. Continuous beam with cantilever part.
- Example 6.3. Yield Line Method. Rectangular plate with uniform load.
- Example 6.4. Yield Line Method. Quadratic plate with "uniform" load.
- Example 6.5. Yield Line Method. Distinguish between possible and impossible mechanisms.

The full texts of the examples are enclosed in the following.

Example 3.1. Serviceability Limit State. Deflections. Rectangular plate with simple supports and uniform loading.

The problem

The quadratic slab shown below is simply supported along all four sides and loaded by a uniform load q.



The quadratic slab

The maximal short-term deflection of the slab needs to be checked for two load combinations

1. Low load: The deadload g plus a uniform load q of 10 kN/m^2
2. High load: The deadload g plus a uniform load q of 40 kN/m^2

The deflections depend significantly on whether the slab is cracked, as an uncracked slab will have far less deflection than a cracked slab.

We will therefore initially assume that the slab is

1. uncracked at ”low” loads and
2. cracked at ”high” load, but still in the linear elastic range of the materials

We must therefore estimate the deflections and also verify that our assumptions are correct.

We will in the end compare the estimated deflections to experimental results in order to see how these assumptions influences our estimates and in order to see the correlation between the estimates and the experimental behaviour – although the dimensions of the slab are smaller than those of more realistic slabs (it was designed for lab testing).

Geometry and material parameters

The reinforced concrete slab has a span length L = 2 m and is h = 81 mm thick and is reinforced with steel bars Ø7 mm per 100 mm in each direction, with an effective height d = 66mm.

The material properties are

$$f_{cm} = 25MPa$$
$$f_{ck} = 20MPa$$
$$f_{yk} = 400MPa$$
$$f_{uk} = 440MPa$$

The modulus of elasticity are estimated as

$$E_{cm} = 22000\left(\frac{f_{cm}}{10}\right)^{0,3} = 22000\left(\frac{25}{10}\right)^{0,3} = 27085MPa$$

The slabs will normally form visible cracks when the tensile stresses exceed the tensile strength after which the slabs will have a decreased stiffness. The tensile strength f_{ctm} is estimated as

$$f_{ctm} = 0,3 f_{ck}{}^{2/3} = 0,3 \cdot 20^{2/3} = 2,21MPa$$

Case 1: Low load

The slabs own weight g is estimated from its density incl. reinforcement of 24 kN/m^2, leading to

$$g = 24 \cdot 0,081 = 1,94 kN/m^2$$

We will assume that the load is so low that the slab is still uncracked. In this situation the plate stiffness is estimated as

$$D = \frac{E_{cm} h^3}{12(1-\upsilon^2)} = \frac{27085 \cdot 81^3}{12(1-0,2^2)} = 1249,5 \cdot 10^6 \, Nmm^2 / mm$$

The deflection may now be determined from Figure 3.3b as

$$u_{\max} = \alpha \frac{pL^4}{D} = \alpha \frac{(g+q)L^4}{D} = 0,0047 \frac{(10+1,94) \cdot 10^{-3} \cdot 2000^4}{1249,5 \cdot 10^6} = 0,8 mm$$

We will, however, need to check whether the assumption of an uncracked slab is correct and we do that be estimating the maximal tensile stress as

$$\sigma_{t,\max} = 6 \frac{m_{\max}}{h^2} = 6 \frac{\beta_1 (g+q) L^2}{h^2} = 6 \frac{0,0486 \cdot (10+1,94) \cdot 10^{-3} \cdot 2000^2}{81^2} = 2,12 MPa \leq f_{ctm} = 2,21 MPa$$

We can see that this is below the tensile strength and the slab may therefore correctly be estimated as uncracked. The slab would probably crack at a load of app.

$$q = \frac{3,36}{2,12}(10+1,94) - 1,94 = 16,98 kN/m^2$$

Loads above this level would lead to a rapid decrease of the stiffness.

Case 2: High load

The high load (40+1,94=41,94kN/m^2) is well above the load at which the cracks occur (16,98kN/m^2) and we must therefore assume a cracked cross-section for which we estimate

$$\alpha = E_s / E_{cm} = 2\cdot 10^5 / 27085 = 7,38$$

$$S_t = b\cdot x\cdot(-x/2) + \alpha A_s(d-x) = 1000\cdot x\cdot(-x/2) + 7,38\cdot 10\pi(7/2)^2(66-x) = 0 \Leftrightarrow x = 16,88mm$$

$$I_t = \frac{1}{12}bx^3 + bx\cdot(x/2)^2 + \alpha A_s(d-x)^2$$

$$= \frac{1}{12}1000\cdot 16,88^3 + 1000\cdot 16,88\cdot(16,88/2)^2 + 7,38\cdot 10\pi(7/2)^2(66-16,88)^2$$

$$= 8,742\cdot 10^6 mm^4 / m = 8,742\cdot 10^3 mm^2 / mm \Rightarrow$$

$$EI = E_c I_t = \frac{E_s}{\alpha} I_t = \frac{2\cdot 10^5}{7,38}\cdot 8,742\cdot 10^3 = 236,77\cdot 10^6 Nmm^2 / mm$$

We use this stiffness in the expression for the deflection, where we replace the uncracked D with our cracked EI and find

$$u_{\max} = \alpha\frac{pL^4}{EI} = \alpha\frac{(q+g)L^4}{EI} = 0,0047\frac{(40+1,94)\cdot 10^{-3}\cdot 2000^4}{236,77\cdot 10^6} = 13,32mm$$

We have, however, based our estimations on the assumption, that the materials are still in the linear area and we will therefore need to estimate the maximal tensile stress in the reinforcement and the maximal compressive stress in the concrete. This requires first the maximal bending moment estimated as

$$m_{\max} = \beta_1(g+q)L^2 = 0,0486\cdot(40+1,94)\cdot 10^{-3}\cdot 2000^2 = 8,154\cdot 10^3 Nmm/mm$$

and then the stresses estimated as

$$\sigma_{c,\max} = \frac{m_{\max}}{I_t}x = \frac{8,154\cdot 10^3}{8,742\cdot 10^3}16,88 = 15,745MPa < f_{ck} = 20MPa$$

$$\sigma_{s,\max} = \alpha\frac{m_{max}}{I_t}(d-x) = 7,38\cdot\frac{8,154\cdot 10^3}{18,742\cdot 10^3}(66-16,88) = 345MPa < f_{yk} = 400MPa$$

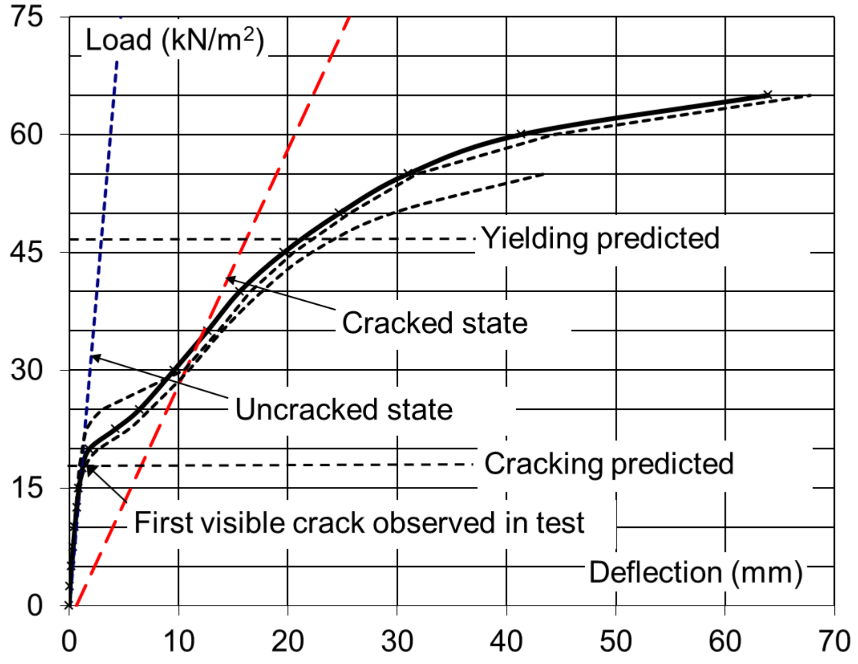
This shows us that the stresses in the concrete and in the steel are well below their maximal possible values. It also indicates that the reinforcement would start to yield at a load of

$$q = \frac{400}{345}(40+1,94) - 1,94 = 46,7kN/m^2$$

Loads above this level are possible (as $f_{uk} = 440MPa > f_{yk} = 400MPa$), but will lead to large deflections (as the reinforcements strain will grow rapidly due to the yielding).

Comparison with tests

Three identical slabs were cast and tested at Stuttgart University by Bach and Graf from 1915 with this design and materials (you will meet that test again later, when we deal with the yield line method and the crack formations and you may see the development of cracks and deformations by clicking this link).



Comparison between the three tested slabs and the estimated stiffnesses, cracking load and yielding load.

We observe a fair correlation between the experimental results and our estimations of uncracked stiffness and cracking load, but also large differences between the uncracked and cracked stiffness.

As this comparison shows, the estimation of the deflection will be quite conservative in a large load range, when we assume that all the cross-sections are cracked. However, a more precise estimation of the deflection of the partly cracked slab may be possible with a non-linear FEM-modeling.

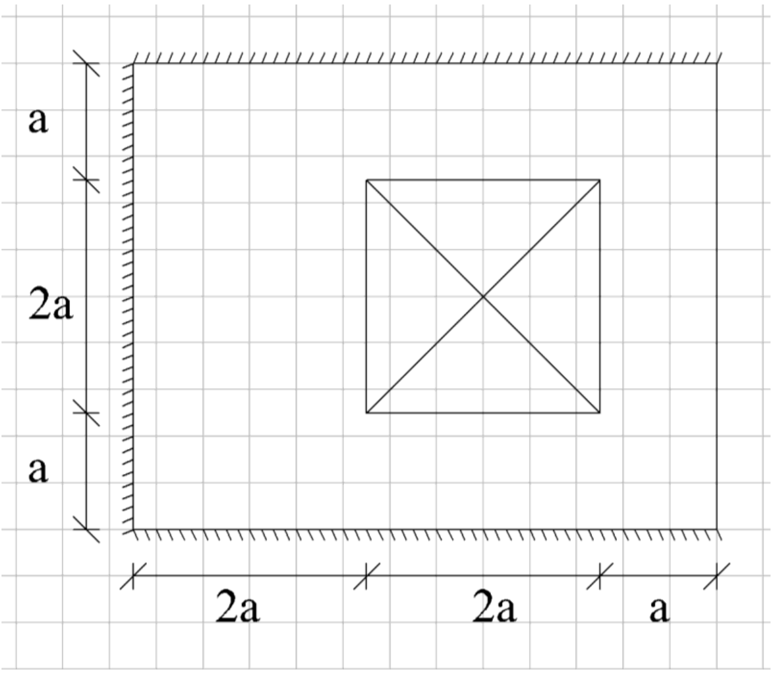
References

Bach, C. and Graf, O.: ”Tests with simply supported, quadratic reinforced concrete plates” (In German:” Versuche mit allseitig aufliegenden, quadratischen und rechteckigen eisenbetonplatten”), Deutscher Ausshuss für Eisenbeton, Heft 30, Berlin 1915.

Example 3.2. Serviceability Limit State. Deflections.
Rectangular plate with hole and three sides supported.

The problem

A rectangular slab with a hole is simply supported along all three sides and is loaded with its own weight g and a uniform load q.



The rectangular slab (a = 1 m)

The maximal short-term deflection of the slab needs to be checked for two load situations

3. Short term load: The deadload g plus a uniform load q of 15 kN/m^2
4. Long term load: The deadload g plus a uniform load q of 7 kN/m^2

and we need to verify that the deflection is less than L/250, where L is the shortest side of the slab.

Geometry and material parameters

The reinforced concrete slab has the dimension of 4 by 5 m with a 2 by 2 m hole. The slab is h = 170 mm thick and is reinforced with steel bars Ø10 mm per 150 mm in each direction, with an effective height d = 140 mm.

The material properties are

$$f_{ck} = 35MPa$$
$$f_{cm} = f_{ck} + 8 = 43MPa$$
$$f_{yk} = 550MPa$$

The modulus of elasticity are estimated as

$$E_{cm} = 22000\left(\frac{f_{cm}}{10}\right)^{0,3} = 22000\left(\frac{43}{10}\right)^{0,3} = 34077MPa$$

The estimated modulus of elasticity is the short-term value used for estimating short-term deformations, whereas a long-term value must be used for estimating long-term deflections. This modulus is estimated by reducing the short term modulus by a factor of $1+\varphi$, where φ is the creep factor, which can be set to 3 for most applications.

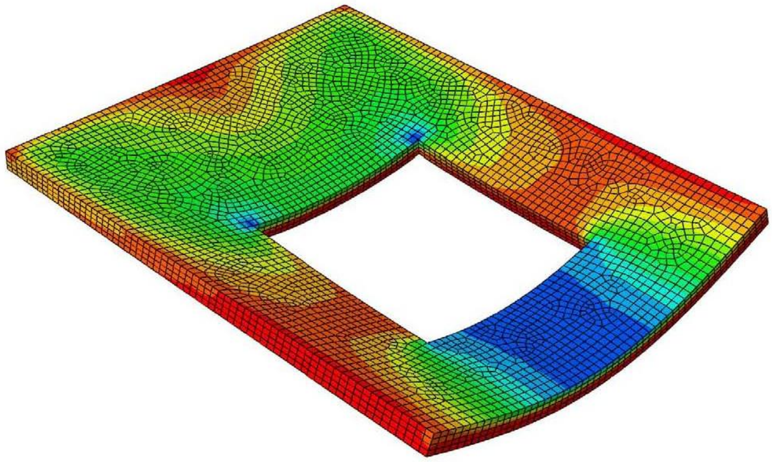
The slabs will normally form visible cracks when the tensile stresses exceed the tensile strength after which the slabs will have a decreased stiffness. The tensile strength f_{ctm} is estimated as

$$f_{ctm} = 0,3 f_{ck}{}^{2/3} = 0,3 \cdot 35^{2/3} = 3,21MPa$$

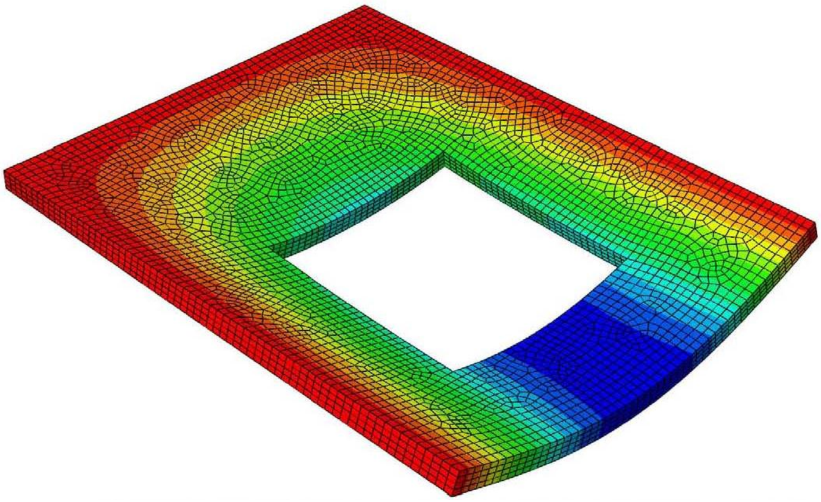
Structural analysis of the slab (FEM)

The deflections are difficult (or impossible) to estimate with simple analytical methods. This is the case, even if we assume that the slab is uncracked (which may be somewhat unsafe and predict too low deflections) or that all the cross-sections in the slab are fully cracked (which may be too conservative and lead to too high estimate of the deflections).

We may, however, carry out a linear elastic FEM-analysis using e.g. the Abaqus program or any other program and we will the find the maximal main stress and the maximal deflection for a certain stiffness D and a certain load p.



Main stresses in Pa for $p=100\text{kN/m}^2$ (maximal tensile stress is 38,25MPa).



Deflection in m for $p=100\text{kN/m}^2$ and $D=1540\cdot10^6\text{Nmm}^2/\text{mm}$ (maximal deflection is 23,95 mm)

We do know that the load level used in the FEM-analysis does not correspond to our load levels, nor will the slabs stiffnesss in the FEM-analysis correspond to our values. The stresses are , however, proportional to p and the deflections are proportional to p/D and we will therefore be able to use the FEM-results for our slab by scaling these results.

Case 1: Short term load

The slabs own weight g is estimated from its density incl. reinforcement of 24 kN/m^2, leading to

$$g = 24 \cdot 0{,}170 = 4{,}08 kN/m^2$$
$$p = q + g = 15 + 4{,}08 = 19{,}08 kN/m^2$$

We may check if the tensile stress in the uncracked state exceeds the flexural strength by using the result from the FEM-analysis

$$\sigma_{c,tension} = 38{,}25 \frac{19{,}08}{100} = 7{,}30 MPa > f_{ctm} = 3{,}21 MPa$$

We must therefore estimate the slabs stiffness in the cracked state as

$$\alpha = E_s / E_{cm} = 2 \cdot 10^5 / 34077 = 5{,}87$$
$$S_t = b \cdot x \cdot (-x/2) + \alpha A_s (d - x) = 1000 \cdot x \cdot (140 - x/2) + 5{,}87 \cdot \frac{1000}{150} \pi (10/2)^2 (140 - x) = 0$$
$$\Leftrightarrow x = 26{,}42 mm$$
$$I_t = \frac{1}{12} b x^3 + b x \cdot (x/2)^2 + \alpha A_s (d - x)^2$$
$$= \frac{1}{12} 1000 \cdot 26{,}42^3 + 1000 \cdot 26{,}42 \cdot (26{,}42/2)^2 + 5{,}87 \cdot \frac{1000}{150} \pi (10/2)^2 (140 - 26{,}42)^2$$
$$= 45{,}791 \cdot 10^6 mm^4 / m = 45{,}791 \cdot 10^3 mm^2 / mm \Rightarrow$$
$$EI = E_c I_t = \frac{E_s}{\alpha} I_t = \frac{2 \cdot 10^5}{5{,}87} \cdot 45{,}791 \cdot 10^3 = 1560{,}4 \cdot 10^6 Nmm^2 / mm$$

which leads us to the estimate of the deflection as

$$u_{\max} = 23{,}95 \frac{19{,}08}{100} \cdot \frac{1540 \cdot 10^6}{1560 \cdot 10^6} = 4{,}5 mm < L/250 = 4000/250 = 16 mm$$

Case 2: Long term load

The load in this situation is

$$q = q + g = 7 + 4,08 = 11,08 kN / m^2$$

and we estimate the EI as for the short term load, but with a different modulus

$$\alpha = E_s / E_{cm} = 2 \cdot 10^5 / (34077 / 4) = 23,48$$

$$S_t = b \cdot x \cdot (-x / 2) + \alpha A_s (d - x) = 1000 \cdot x \cdot (140 - x / 2) + 23,48 \cdot \frac{1000}{150} \pi (10 / 2)^2 (140 - x) = 0$$

$$\Leftrightarrow x = 47,65 mm$$

$$I_t = \frac{1}{12} b x^3 + b x \cdot (x / 2)^2 + \alpha A_s (d - x)^2$$

$$= \frac{1}{12} 1000 \cdot 47,65^3 + 1000 \cdot 47,65 \cdot (47,65 / 2)^2 + 23,48 \cdot \frac{1000}{150} \pi (10 / 2)^2 (140 - 47,65)^2$$

$$= 140,90 \cdot 10^6 mm^4 / m = 140,90 \cdot 10^3 mm^2 / mm \Rightarrow$$

$$EI = E_c I_t = \frac{E_s}{\alpha} I_t = \frac{2 \cdot 10^5}{23,48} \cdot 140,90 \cdot 10^3 = 1200 \cdot 10^6 Nmm^2 / mm$$

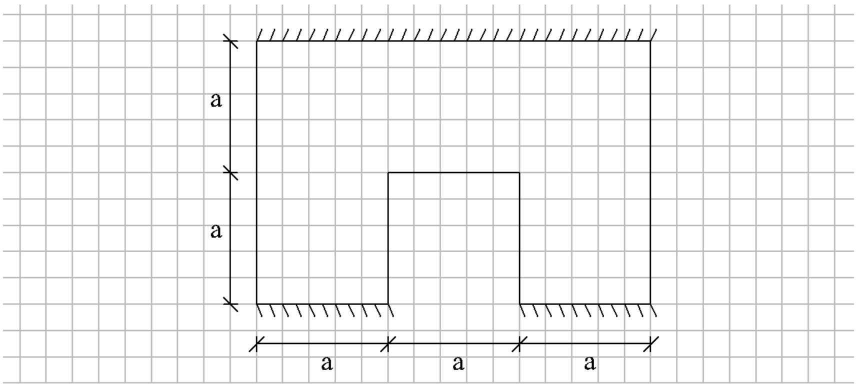
which leads us to the estimate of the deflection as

$$u_{\max} = 23,95 \frac{11,08}{100} \cdot \frac{1540 \cdot 10^6}{1200 \cdot 10^6} = 3,4 mm < L / 250 = 4000 / 250 = 16 mm$$

Example 5.1. C-shaped plate with some simply supported sides and some free sides.

The problem

A Strip Method model needs to be developed for a rectangular reinforced concrete plate with a hole and simply supported on three sides as shown below.

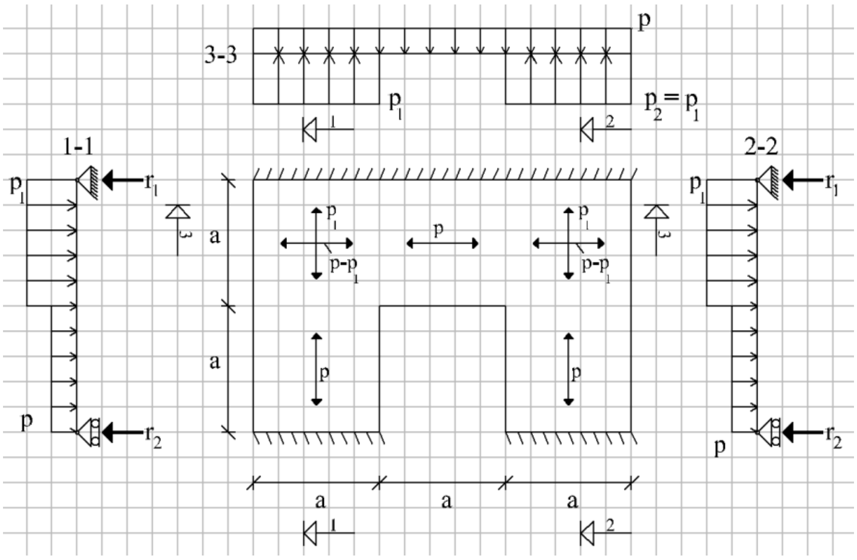


The plate geometry

The plate has the same ultimate bending moment capacity in all both directions (x and y) and for both positive (m_{ux} and m_{uy}) and negative moments ($m_{ux}^{'}$ and $m_{uy}^{'}$):

$$m_{ux} = m_{uy} = m_{ux}^{'} = m_{uy}^{'} = m_u$$

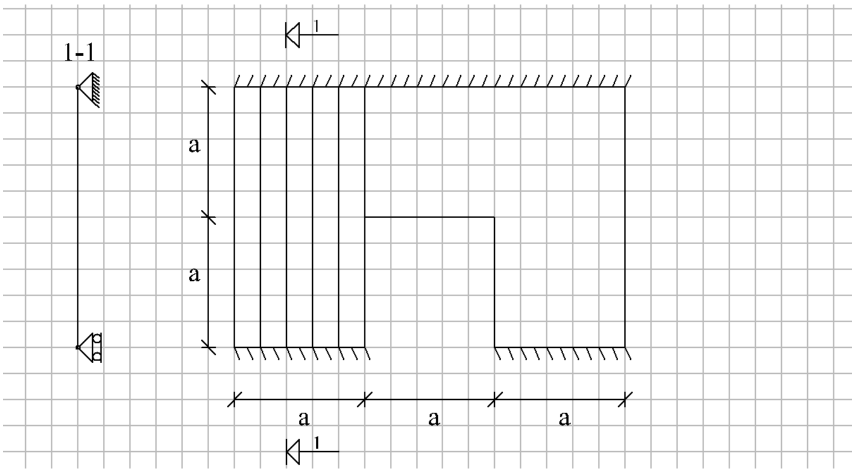
The lower limit p$^-$ for the load-carrying capacity of this plate shall be estimated for an uniform load p over the plate and we do therefore set up a model as shown below:



The model of the strips in the plate.

Please note that this drawing of the model for all the strips along with the plan of the plate with the loads indicated on is a very good way of checking that the models correspond to the load-distribution ion the plan and that the loads transferred in the two directions actually add up to the full load.

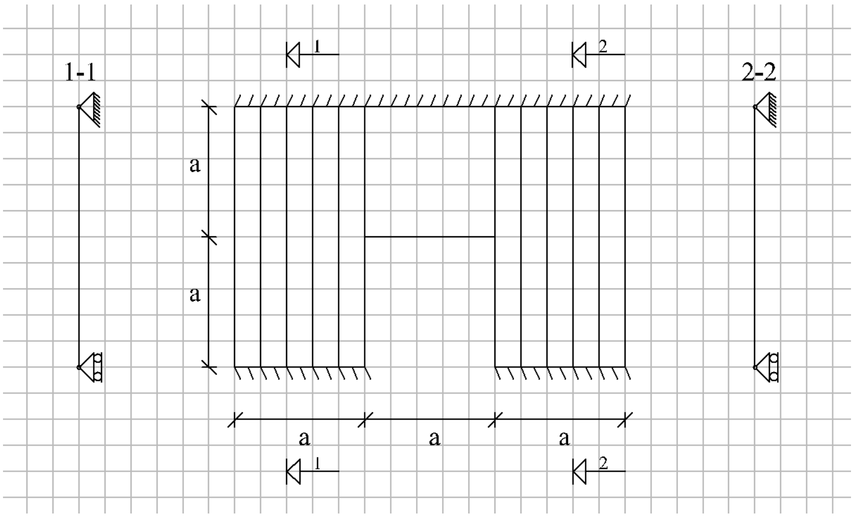
Developing the strip model



Step 1:

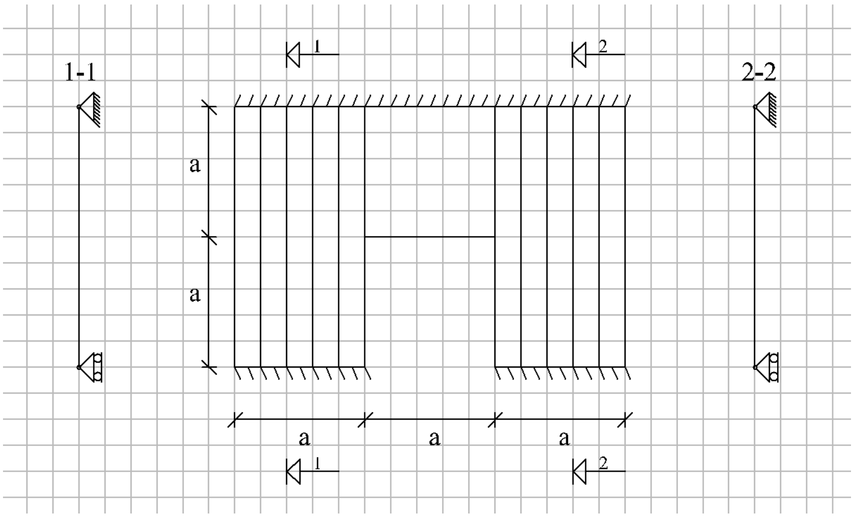
A number of strips (similar to beams, floorboards, planks and similar) must be placed an cover all the plates area.

We start therefore with strip 1, which is placed from one simple support to another.



Step 2:

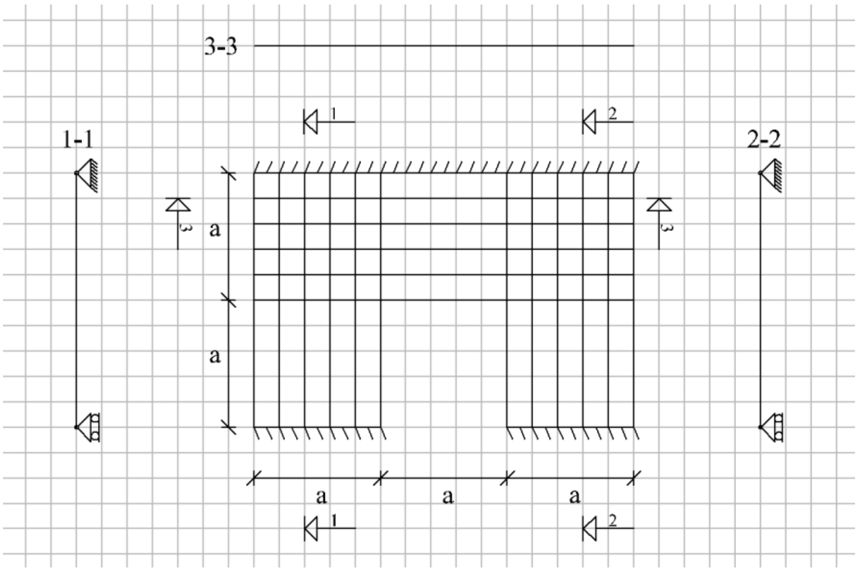
We continue with another set of strips (no. 2), also placed from one simple support to the next.



Step 3:

We must now place a new layer of strips (no 3) on top of strips 1 and 3.

This layer of strips is not directly supported, but rests on top of the strips 1 and 2.

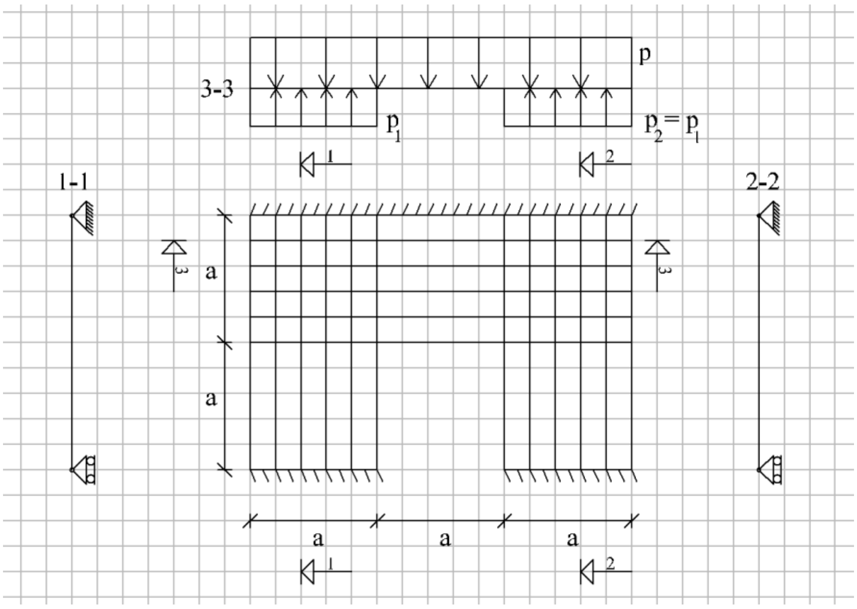


Step 4:

We must now start to put the loading on the strips and to decide how the strips are supported (directly or by the other strips).

We must always start with the last strip (as we place the loading on the uppermost layer of strips)

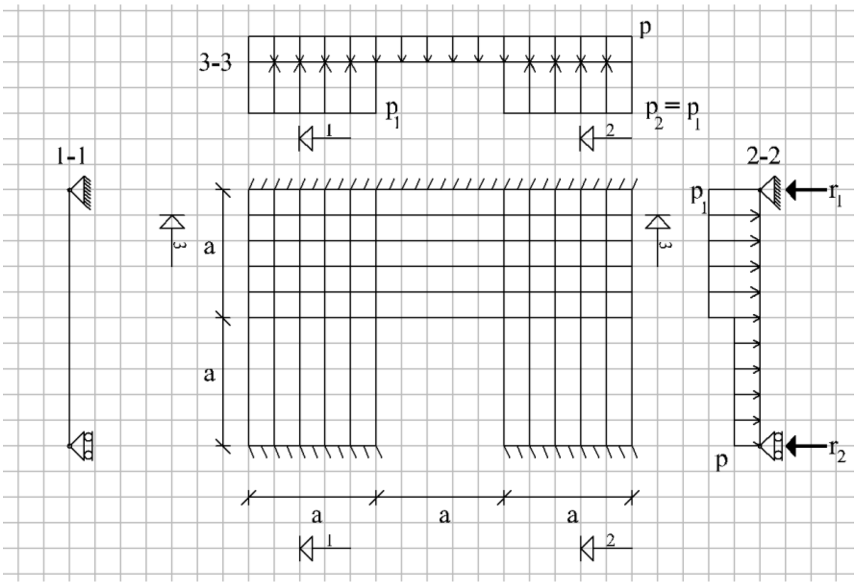
So we start with strip 3.



Step 5: The strip 3 is not supported at the ends, but rest on strips 1 and 2.

The strip 3 rests on top of strip 1 and 2, which may press up with a distributed reaction or force. These are unknown and are therefore named p_1 and p_2.

The symmetry means, however, that p_2=p_1. Please note that a symmetrical problem has normally a symmetrical solution – and that symmetry reduces the calculations later.

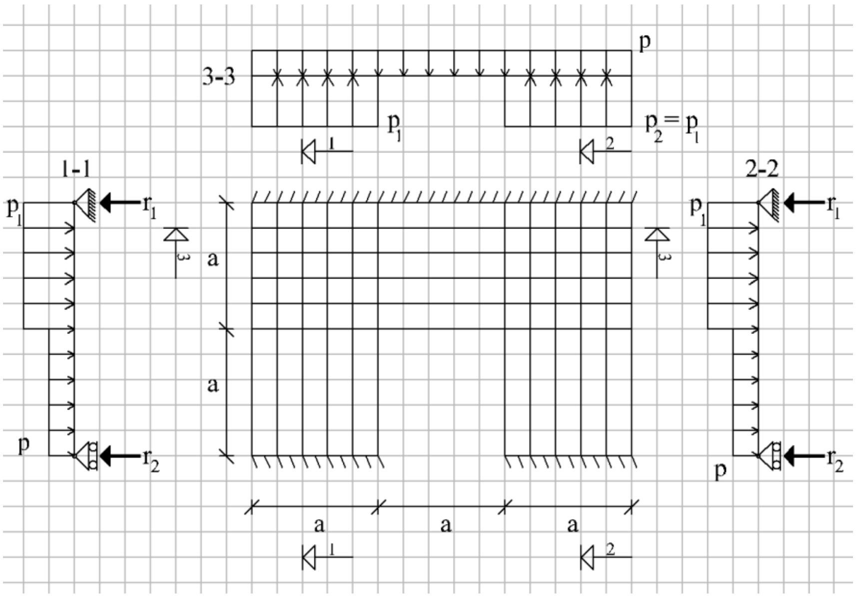


Step 6:

We see that the strip 3 transfer a distributed reaction or load p_1 to the strip 2 in the area, where the strip 2 is covered by strip 3.

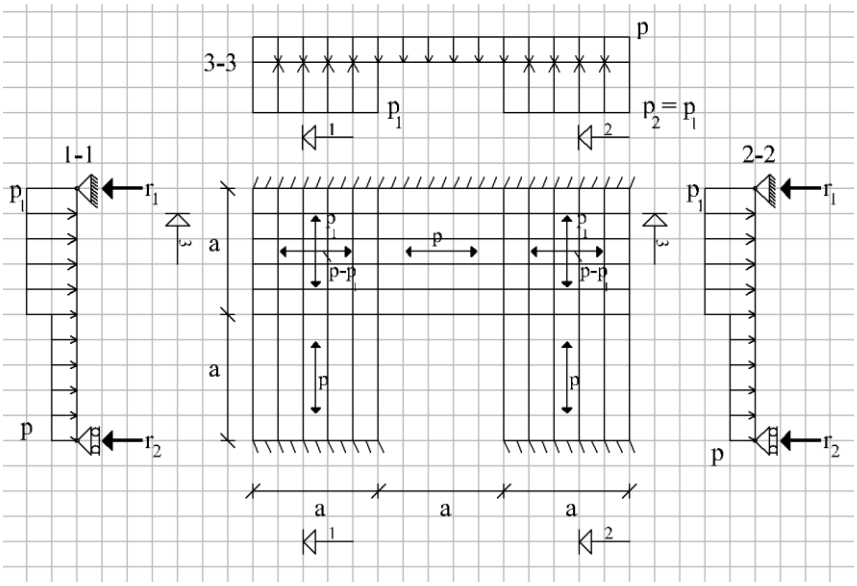
The area, not covered by strip 3, is loaded with the uniform load p.

The strip has two unknown reactions, here named as r_1 and r_2.



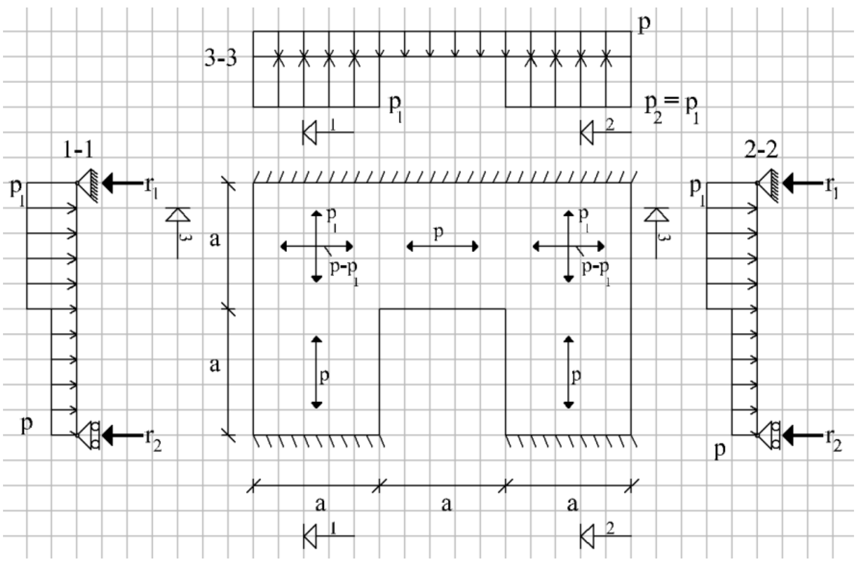
Step 7:

We see from the symmetry, that the model for strip 1, is the same as for strip 2.



Step 8:

As the last part of our development of the model, we indicate on the drawing how the strips in the two directions share the uniform load p.



Step 9: We remove the lines, which indicated the strips on the drawing and have now established our model.

This setup of a model is widely used in the literature – whereas the drawing of the actual lines for the strips is normally left out (but they are a good tool in the learning process)

Calculating the lower limit solution

Each strip need to be checked for moment equilibrium (so the strip does not tilt) and for vertical equilibrium.

This must be checked on the last placed strip, as this strip is placed on top of all the other strips, after which the second last strip is checked etc.

This corresponds to the approach we normally use for structures: We normally build the structure starting from the bottom – but we normally calculate the forces from the top and work our way down.

We will thus estimate strip 3 first, then strip 2 and at last strip 1.

<u>Strip 3:</u>

Moment equilibrium: The use of symmetry secures the equilibrium. If we had not used p1=p2 from the beginning, then this control would have determined that the two reactions/loads were identically the same.

Vertical equilibrium: $3pa = p_1 a + p_2 a = 2p_1 a \Leftrightarrow p_1 = \frac{3}{2} pa$

The maximal moment can be seen to be at the midpoint of the strip, where it must be less than or equal to the bending moment capacity

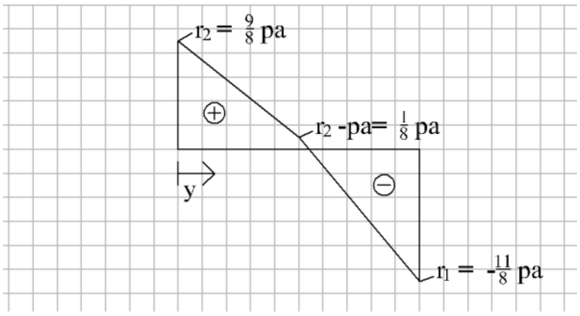
$$m_{3,\max} = -p\frac{3}{2}a \cdot \frac{3}{4}a + p_1 a \cdot a = \frac{3}{8} pa^2 \le m_u \Leftrightarrow p_3^{(-)} = \frac{8}{3}\frac{m_u}{a^2}$$

Strip 2:

Moment equilibrium: $m = -pa \cdot \frac{a}{2} - p_1 a \cdot \frac{3}{2}a + r_1 \cdot 2a \Leftrightarrow r_1 = \left(pa \cdot \frac{a}{2} + p_1 a \cdot \frac{3}{2}a \right) / \, 2a = \frac{11}{8}pa$

Vertical equilibrium: $r_1 + r_2 = pa + p_1 a \Leftrightarrow r_2 = pa + p_1 a - r_1 = \frac{9}{8}pa$

The strip will have its maximal moment(s) at the position(s), where the shear force is equal to zero, which is simplest found by drawing the shear force curve as follows



It can be seen that the shear force has its zero at a < y < 2a, where

$$v = r_2 - pa - p_1(y-a) = 0 \Rightarrow y = \frac{r_2 - pa}{p_1} + a = \frac{\frac{9}{8}pa - pa}{\frac{3}{2}p} + a = \frac{13}{12}a$$

The maximal moment is estimated in this point as

$$m_{2,\max} = r_2 \cdot \frac{13}{12}a - pa \cdot \left(\frac{13}{12}a - \frac{1}{2}a \right) - p_1 \frac{1}{12}a \cdot \frac{1}{2 \cdot 12}a = \frac{121}{192}pa^2 \le m_u \Leftrightarrow p_2^{(-)} = \frac{192}{121}\frac{m_u}{a^2}$$

Strip 1:

The strip 1 is identical to the strip 2 in both loads, reactions and strength so

$$p_1^{(-)} = p_2^{(-)}$$

The load carrying capacity:

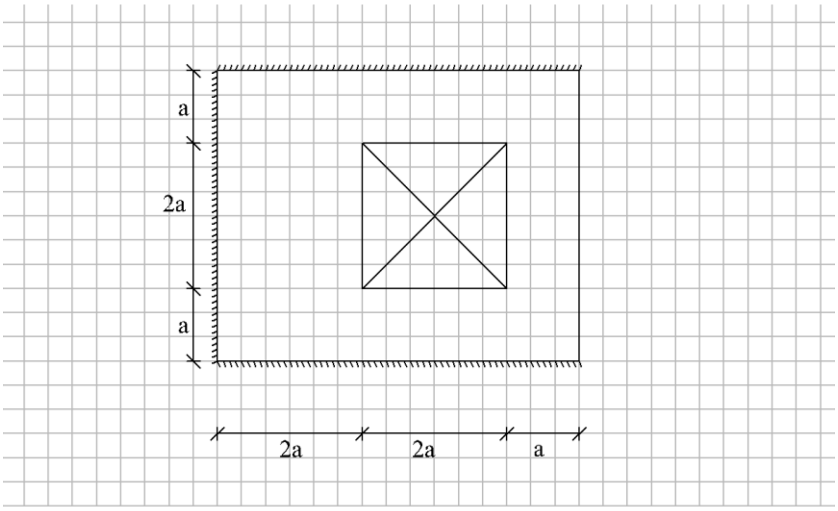
The load carrying capacity of the plate is the lowest of the three estimated capacities as

$$p^{(-)} = \text{minimum}\left(p_1^{(-)}, p_2^{(-)}, p_3^{(-)} \right) = \text{minimum}\left(\frac{192}{121}\frac{m_u}{a^2}, \frac{192}{121}\frac{m_u}{a^2}, \frac{8}{3}\frac{m_u}{a^2} \right) = \frac{192}{121}\frac{m_u}{a^2}$$

Example 5.2. Strip Method Design.
Rectangular plate with a hole and three sides supported.

The problem

A Strip Method model needs to be developed for a rectangular reinforced concrete plate with a hole and simply supported on three sides as shown below.



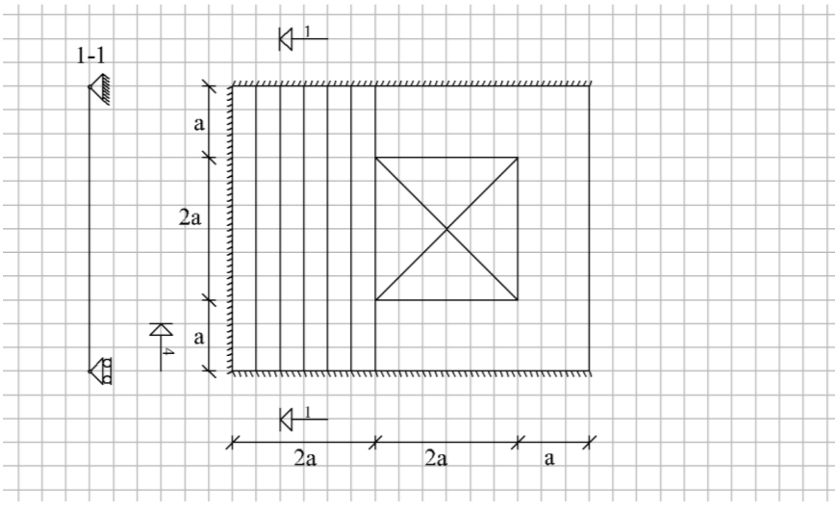
The plate geometry.

The plate has the same ultimate bending moment capacity in both directions and for both positive and negative moments:

$$m_{ux} = m_{uy} = m_{ux}^{'} = m_{uy}^{'} = m_u$$

The lower limit p$^-$ for the load-carrying capacity of this plate shall be estimated for a uniform load p over the plate.

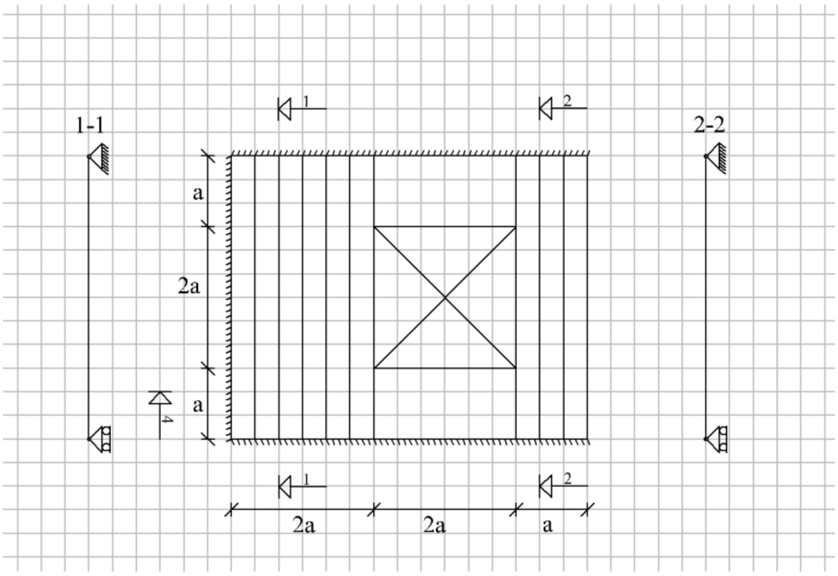
Developing the strip model



Step 1:

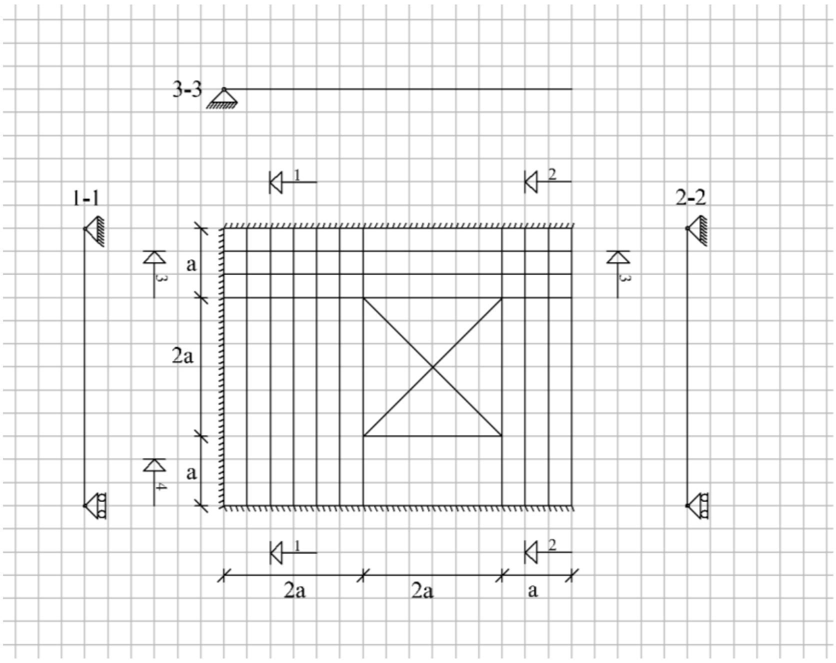
A number of strips (similar to beams, floorboards, planks and similar) must be placed an cover all the plates area.

We start therefore with strip 1, which is placed from one simple support to another.



Step 2:

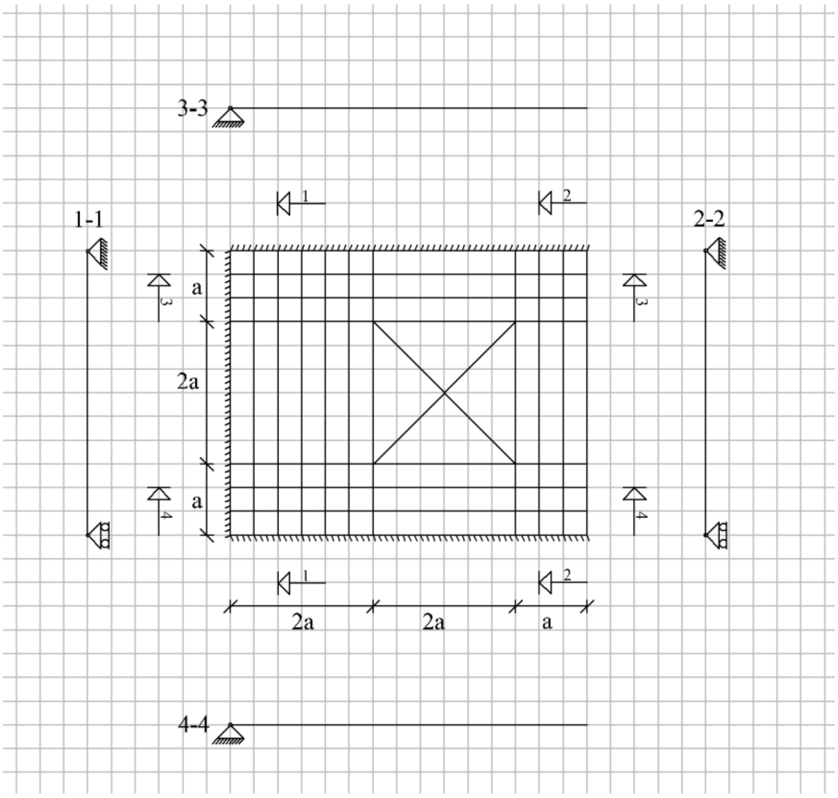
We continue with another set of strips (no. 2), also placed from one simple support to the next.



Step 3:

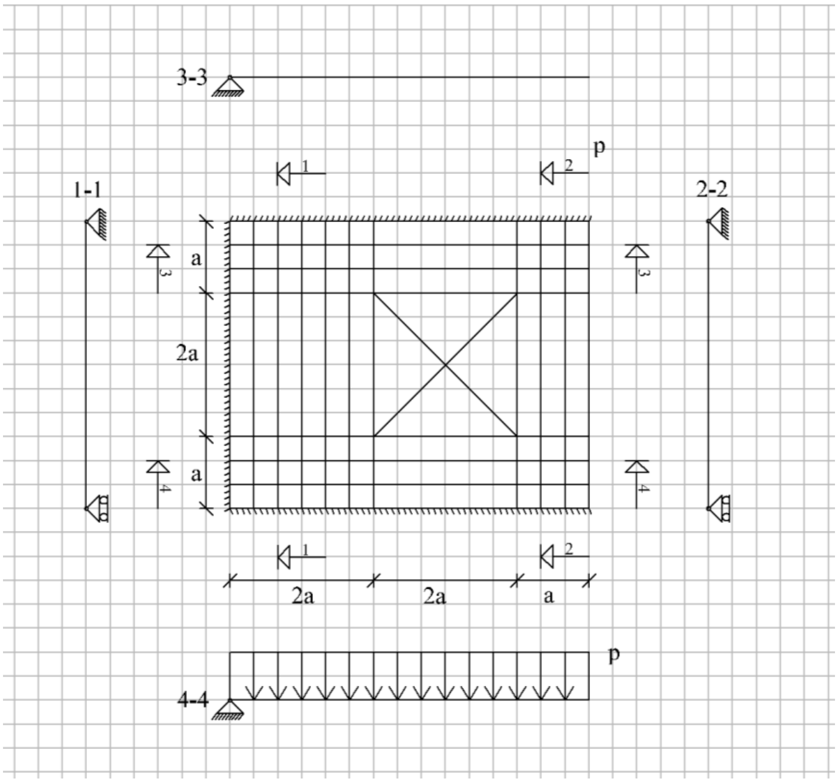
We must now place a new layer of strips (no 3) on top of strips 1 and 3.

This layer of strips is not directly supported, but rests on top of the strips 1 and 2.



Step 4:

We continue in the same manner with a layer of strips (no 4) on top of strips 1 and 2.

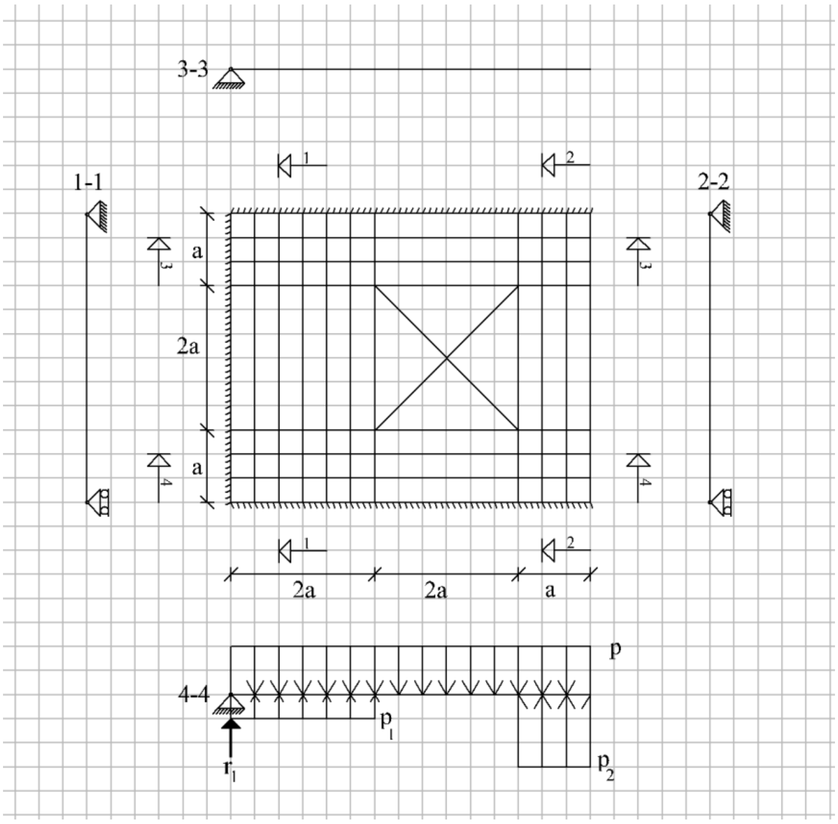


Step 5:

We must now start to put the loading on the strips and to decide how the strips are supported (directly or by the other strips).

We must always start with the last strip (as we place the loading on the uppermost layer of strips)

So we start with strip 4.

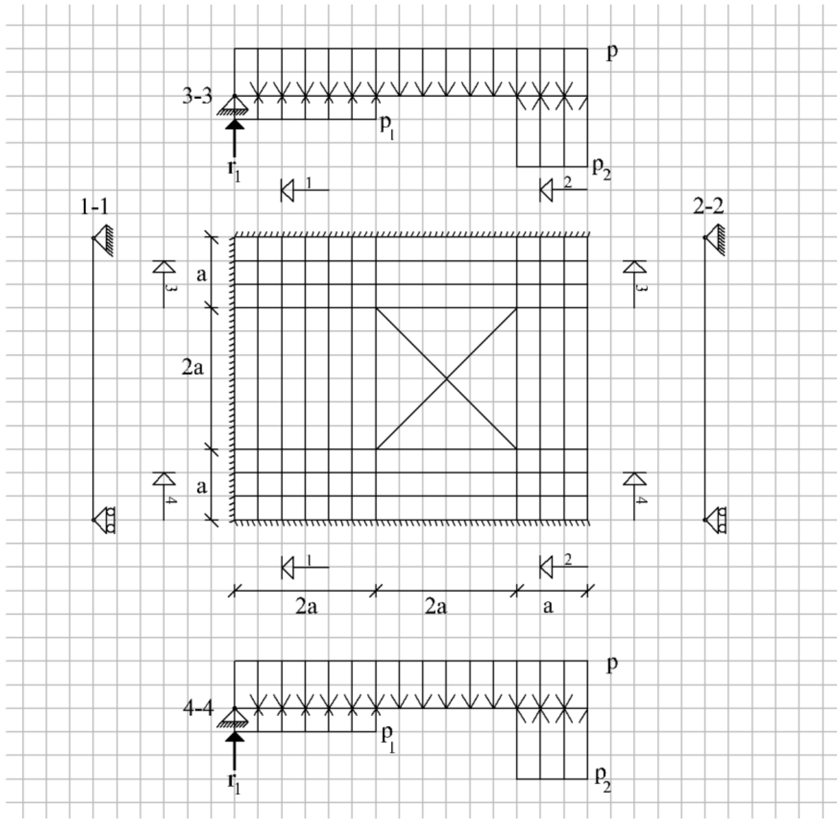


Step 6:

The strip 4 is simply supported at the end, but rest also on strips 1 and 2.

We name the unknown reaction r_1 (and later reactions r_2, r_3 etc.).

The strip 4 rests on top of strip 1 and 2, which may press up with a distributed reaction or force. These are unknown and are therefore named p_1 and p_2..

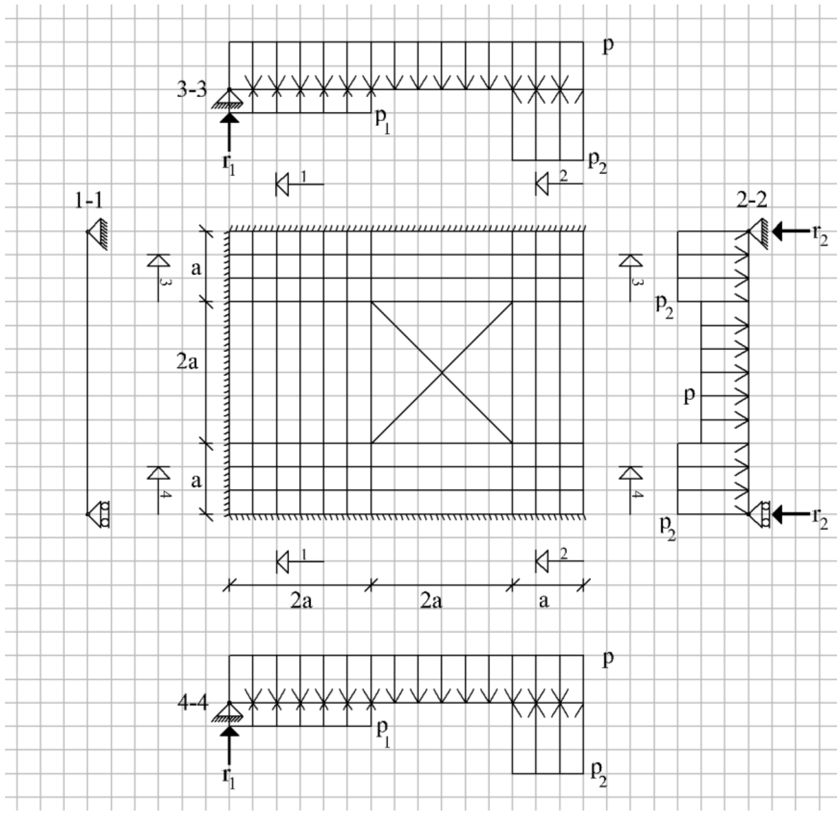


Step 7:

We continue with strip 3.

The plate is symmetrical and loaded by a uniform load. We see therefore that the strip 3 should be the same as strip 4 and we use this information.

Note that taking the symmetry into account reduce the amount of calculations we have to carry out – but note also that the most optimal solutions to symmetrical problems tend to be symmetrical.

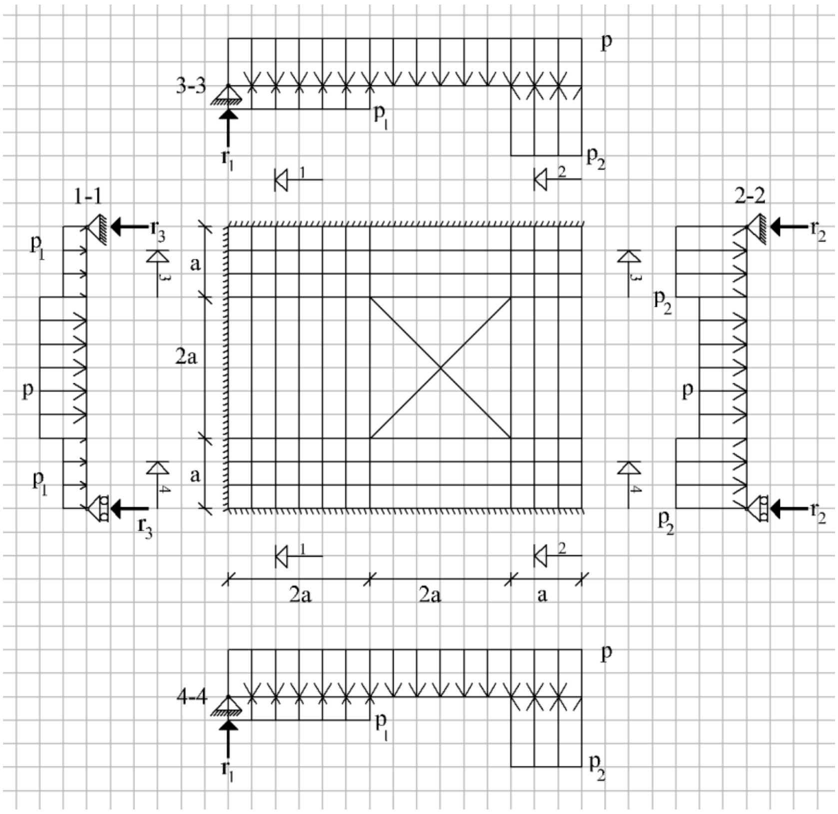


Step 8:

We see that the strips 3 and 4 load the strip 2 with a load p_2 in the areas, where the strip 2 is covered by the strips 3 ad 4.

The strip 2 is directly loaded by p in the middle, where no other strips are placed above this strip.

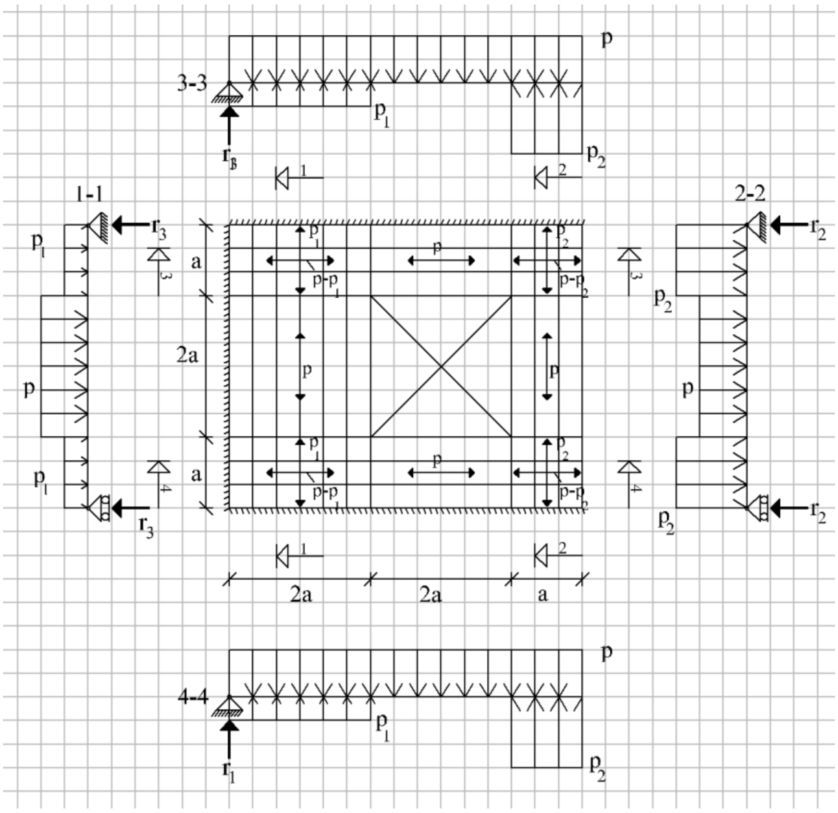
The strip is symmetrical with a symmetrical load and the unknown reactions at the ends are therefore identical and we name them r_2.



Step 9:

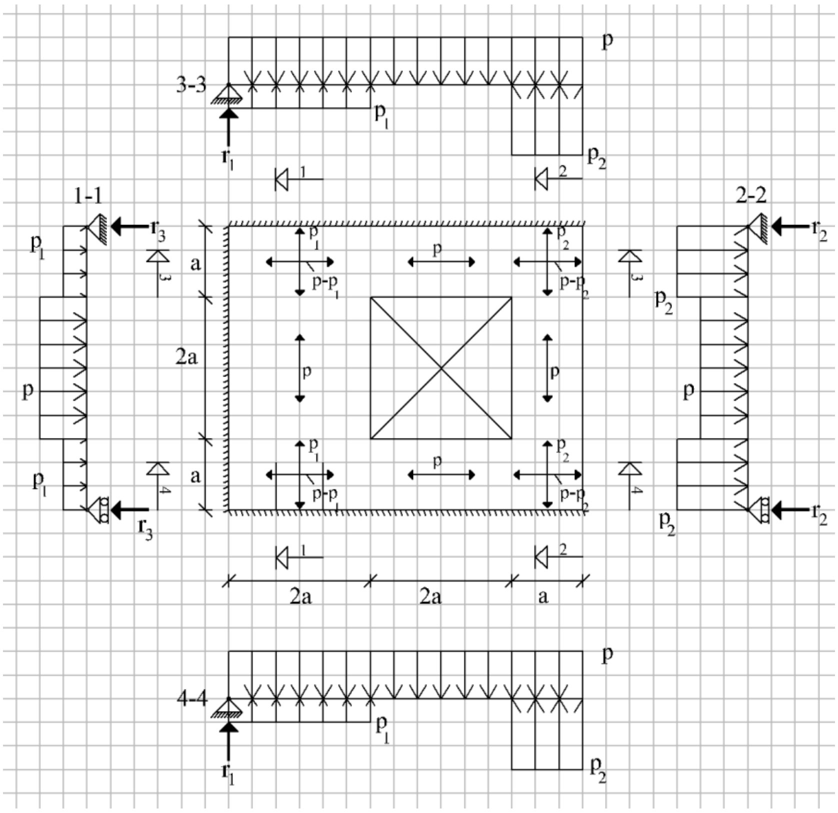
The last strip, strip 1, is loaded in the same manner as strip 2, although the loads from strip 3 and 4 are denoted p_1.

The unknown reactions at the end of the symmetrically loaded and supported strip are named r_3.



Step 10:

As the last part of our development of the model, we indicate on the drawing how the strips in the two directions share the uniform load p.



Step 11: We remove the lines, which indicated the strips on the drawing and have now established our model.

This setup of a model is widely used in the literature – whereas the drawing of the actual lines for the strips is normally left out (but they are a good tool in the learning process)

Please note: This approach with the drawing of the models for all the strips along with the plan of the plate with the loads indicated on is a very good way of checking that the models correspond to the load-distribution ion the plan and that the load transferred in the two directions actually add up to the full load.

Calculating the lower limit solution.

Each strip need to be checked for moment equilibrium (so the strip does not tilt) and for vertical equilibrium.

This must be checked on the last placed strip, as this strip is placed on top of all the other strips, after which the second last strip is checked etc.

This corresponds to the approach we normally use for structures: We normally build the structure starting from the bottom – but we normally calculate the forces from the top and work our way down. So we start with strip 4, then strip 3, then strip 2 and at last strip 1.

Strip 4:

We notice immediately, that we have a strip with 3 unknowns: p_1, p_2 and r_1.

We have only 2 equilibrium conditions, so we have 3-2=1 degrees of freedom in the static system and will therefore have to choose one of the three parameters or to choose that there should be a specific relationship between some of these parameters. We decide to choose

$$p_1 = p/2$$

Please note: This may be a good choice, which leads to a high value of the lower limit solutions, as p1 is active in an area in a corner, supported on two sides, so it would seem logical if equal parts of the load p were carried in the two directions and it might lead to a fairly high value of the load carrying capacity. This is however, just a simple choice and any other choice would be equally valid and only an optimisation with variation of p1 could determine if this is indeed an optimal choice.

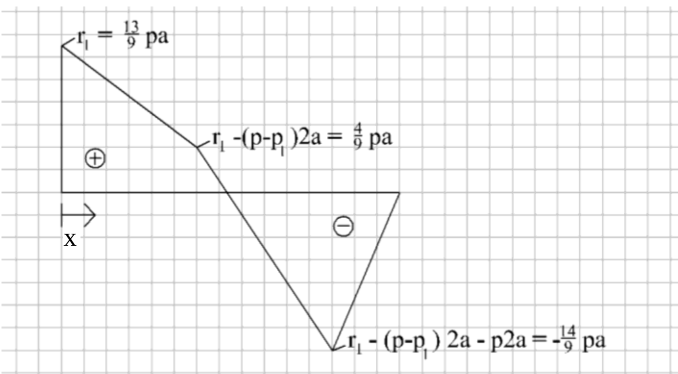
The moment equilibrium requires

$$m = -5pa \cdot \frac{5}{2}a + p_1 2a \cdot a + p_2 a \cdot \frac{9}{2}a = 0 \Leftrightarrow p_2 = \frac{25}{9}p - \frac{4}{9}p_1 = \frac{23}{9}p$$

The vertical equilibrium requires

$$r_1 = 5pa - p_1 2a - p_2 a = \frac{13}{9}pa$$

We may now draw the shear force curve in order to identify the point, where the shear is zero



This means that the shear force is zero at x (between x=2a and x=4a), where

$$v(x) = r_1 - px + 2p_1a = 0 \Rightarrow x = \frac{r_1 + 2p_1a}{p} = \frac{22}{9}a$$

where we find the maximal moment

$$m_{4,\max} = r_1x - p_1 2a \cdot (x-a) - ½px^2 = \frac{161}{81}pa^2 \le m_u \Leftrightarrow p_4^{(-)} = \frac{81}{161}\frac{m_u}{a^2}$$

Strip 3:

The strip 3 is identical to strip 4 in all aspects, so

$$p_3^{(-)} = p_4^{(-)}$$

Strip 2:

We identified the symmetry of the loading and the reactions in strip 2, so the moment equilibrium is fulfilled, but the vertical equilibrium requires then

$$2r_2 = p_2a + 2pa + p_2a \Leftrightarrow r_2 = pa + p_2a = \frac{32}{9}pa$$

The maximal moment is found in the middle of the span as

$$m_{2,\max} = r_2 2a - p_2a \cdot \frac{3}{2}a - pa\frac{a}{2} = \frac{25}{9}pa^2 \le m_u \Leftrightarrow p_2^{(-)} = \frac{9}{25}\frac{m_u}{a^2}$$

Strip 1:

The strip 1 corresponds to strip 2, if one just replaces p_2 and r_2 with p_1 and r_3 and we find therefore

$$r_3 = pa + p_1a = \frac{3}{2}pa$$

$$m_{1,\max} = r_3 2a - p_1a \cdot \frac{3}{2}a - pa\frac{a}{2} = \frac{7}{4}pa^2 \le m_u \Leftrightarrow p_1^{(-)} = \frac{4}{7}\frac{m_u}{a^2}$$

The load carrying capacity

The load carrying capacity of the plate is the lowest of the four estimated capacities as

$$p^{(-)} = \text{minimum}\left(p_1^{(-)}, p_2^{(-)}, p_3^{(-)}, p_4^{(-)}\right) = \text{minimum}\left(\frac{4}{7}\frac{m_u}{a^2}, \frac{9}{25}\frac{m_u}{a^2}, \frac{81}{161}\frac{m_u}{a^2}, \frac{81}{161}\frac{m_u}{a^2}\right) = \frac{9}{25}\frac{m_u}{a^2}$$

Other choises of p_1

Other choices of p_1 would be possible and would lead to other load carrying capacities, either higher or lower.

These other choices may be calculated with the same approach as we have just followed or we may program the solutions into a spreadsheet, Matlab, Matcad or a number of other programs, however, in the case of use of programs, we would normally enter the complete description of the moment curves.

Strip 4 and strip 3:

$$p_2 = \frac{25}{9}p - \frac{4}{9}p_1$$

$$r_1 = 5pa - p_1 2a - p_2 a$$

$$m_4(x) = \begin{cases} r_1 x - ½(p - p_1)x^2 & \text{for} \quad 0 \le x \le 2a \\ r_1 x - (p - p_1)2a(x-a) - ½p(x-2a)^2 & \text{for } 2a < x \le 4a \\ r_1 x - (p - p_1)2a(x-a) - p2a(x-3a) - ½(p - p_2)(x-4a)^2 & \text{for } 4a < x \le 5a \end{cases}$$

$$m_3(x) = m_4(x)$$

Strip 2:

$$r_2 = pa + p_2 a$$

$$m_2(y) = \begin{cases} r_2 y - ½p_2 y^2 & \text{for} \quad 0 \le y \le a \\ r_2 y - p_2 a(y - a/2) - ½p(y-a)^2 & \text{for } a < y \le 3a \\ r_2 y - p_2 a(y - a/2) - 2pa(y-2a) - ½p_2(y-3a)^2 & \text{for } 3a < y \le 4a \end{cases}$$

$$m_2(y = 2a) = \frac{1}{2}(3p + p_2)a^2$$

Strip 1:

$$r_3 = pa + p_1 a$$

$$m_3(y) = \begin{cases} r_3 y - ½p_2 y^2 & \text{for} \quad 0 \le y \le a \\ r_3 y - p_1 a(y - a/2) - ½p(y-a)^2 & \text{for } a < y \le 3a \\ r_3 y - p_1 a(y - a/2) - 2pa(y-2a) - ½p_1(y-3a)^2 & \text{for } 3a < y \le 4a \end{cases}$$

$$m_3(y = 2a) = \frac{1}{2}(3p + p_1)a^2$$

Optimum is reached when the capacity is the same in two of the strips, which happens at $p_1 = p_2$ at which we find

$$p^{(-)} = \frac{39}{96}\frac{m_u}{a^2} = 0,4063\frac{m_u}{a^2}$$

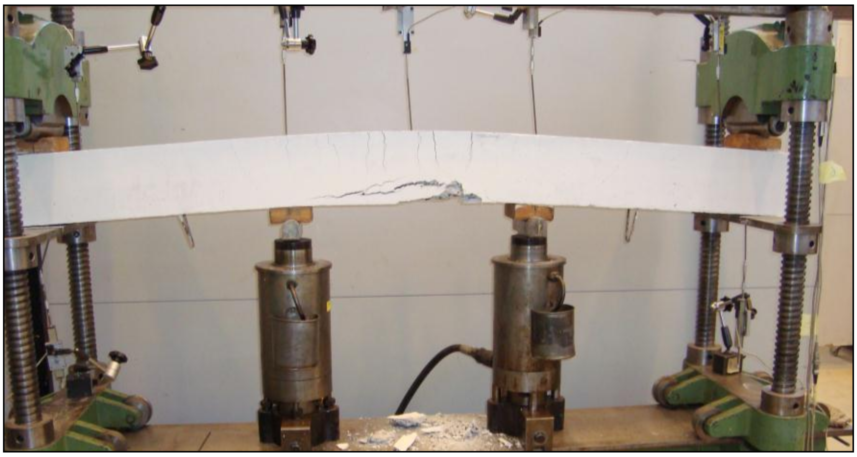
Example 6.1. Yield line method.
Two point loading of a beam – plastic hinge

The problem

We need to try to use the Virtual Works principle to find an upper limit solution for a simply supported beam with the two point loads. The failure mechanism we use will be determined directly from the observed failure of a beam designed, tested and reported in a CDIO project at DTU.

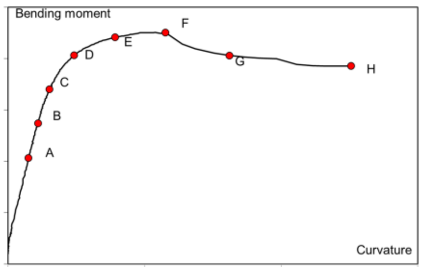
Testing of the beam

The beam is for practical purposes turned up-side down for the testing in the laboratory, so that the tensile zone is at the upper side of the beam. The loading is increased in steps, while the deflections are recorded and cracks are recorded for each load step until the failure load is reached.



The test setup and the failed beam.

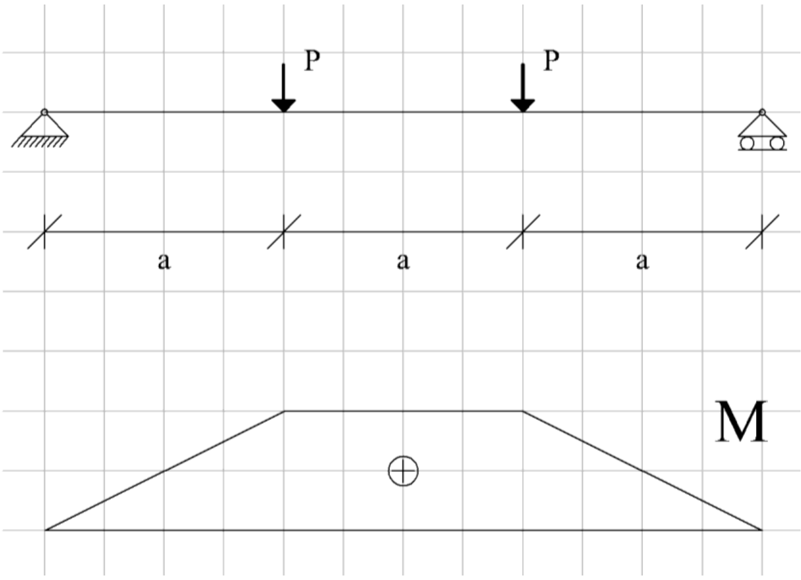
The beam shows yielding and significant crack formation over the length between the two point loads before the failure (as expected from the traditional theories). It is observed that at failure, the deformations grow significantly in the middle third of the beam without any increase of the loads, but with yielding in this area and that the failure is quite ductile and plastic.



The bending moment – curvature relationship
(points A to H correspond to stages in the testing – click on link above the figure).

Static system of the beam

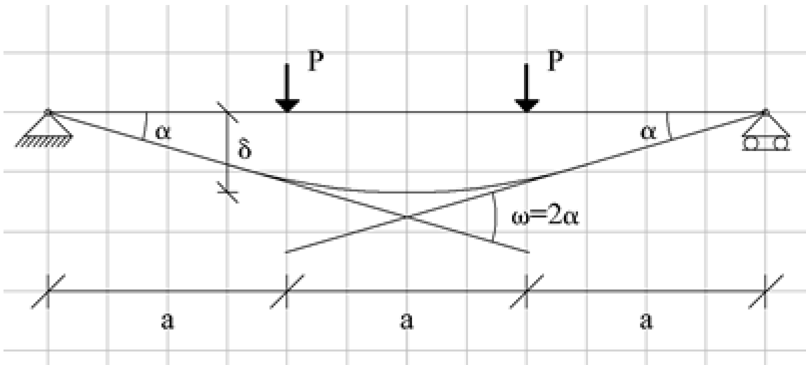
The simply supported beam is loaded by two point loads in the third point positions, which results in a constant bending moment in the middle third of the span as shown below.



The beam and the loads (top) and the resulting bending moment (bottom).

<u>Failure mechanism</u>

The beams failure mechanism is yielding of the reinforcement and crushing of the concrete over a part of the middle third of the beam, which leads to additional deformations as illustrated below.



The failure mechanism for the beam (only incremental deformations in the failure are shown).

The incremental deformation δ shown in the figure is the infinitely small extra deformation, which occur at the exact time and does not correspond to the variations of the deformations before the failure.

Use of the Virtual Works Principle

The internal work is the sum of the bending moments and the incremental curvatures κ_δ in the beam, which occurs at the time, where the failure occurs. This means that

$$W_i = \int M(s) \cdot \kappa_\delta(s) \cdot ds$$

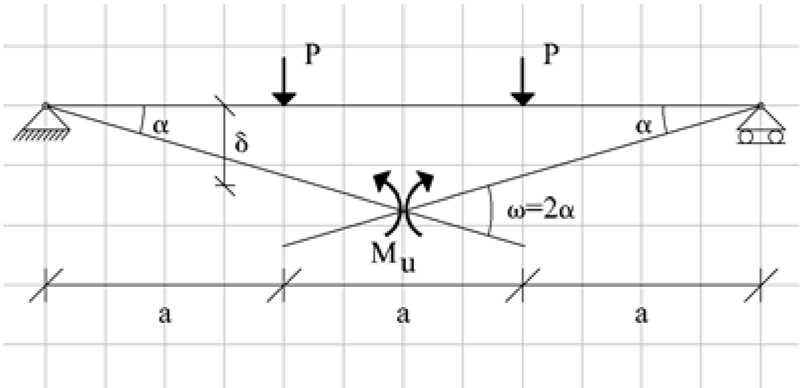
The part of the middle third of the beam yields and bends, whereas rest of the beam do not show any additional curvature. This means that the inner work is calculated as

$$W_i = \int M(s) \cdot \kappa_\delta(s) \cdot ds = \int M_u \cdot \kappa_\delta(s) \cdot ds = M_u \cdot \kappa_\delta \cdot a = M_u \cdot \omega$$

since the bending moment is equal to the positive yielding moment M_u in the middle third of the beam and the bend ω is found as

$$\omega = \int \kappa_\delta(s) \cdot ds$$

This means that the internal work can be estimated as the work in a concentrated plastic hinge with a certain yielding moment and a certain bend. A closer measurement of the failure mechanism will also reveal, that is the incremental bending in the precise moment of failure will be located to one concentrated area in the middle third of the beam.



The concentrated plastic hinge mechanism.

The estimations lead therefore to

$$\omega = 2 \cdot \alpha = 2 \cdot \delta / \left(\frac{3}{2} a\right) = \frac{4}{3}\frac{\delta}{a} \Rightarrow$$

$$W_i = M_u \omega = \frac{4}{3} M_u \frac{\delta}{a}$$

The external work

This is estimated as the sum of the external loads multiplied with the incremental deformations of the beam in the load positions

$$W_e = \int p(s) \cdot \delta(s) \cdot ds + \sum P_i \delta_i = 2 \cdot P \cdot \frac{2}{3} \cdot \delta = \frac{4}{3} P\delta$$

Load-carrying capacity

The upper limit estimate of the load-carrying capacity is calculated by setting the external and the internal work equal to each other as

$$W_i = W_e \Leftrightarrow$$
$$\frac{4}{3} M_u \frac{\delta}{a} = \frac{4}{3} P\delta \Leftrightarrow$$
$$P = M_u / a$$

Comments

This example shows that a plastic hinge may represent a continuous length of the beam with failure (yielding of the cross-section) and that the model for the failure may correctly use such a concentrated plastic hinge, located in a position in the middle of the span.

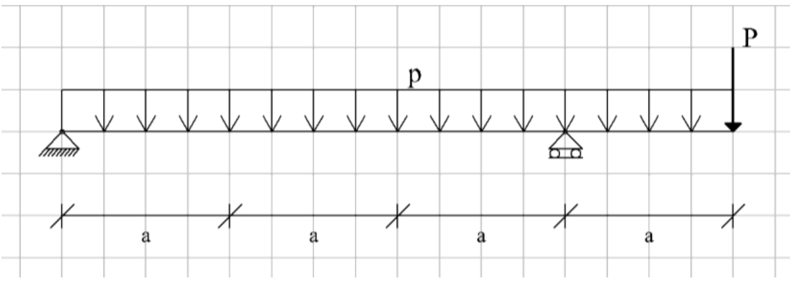
This is the same value as the classic beam theory predicts, since we have used the correct failure mechanism in which case the upper-limit solution will always determine the true capacity. (Use of an incorrect, but possible failure mechanism would normally lead to a predicted capacity above the true value, thus the term upper limit solution)

Notice that this has shown, that the use of an upper limit solution based on an assumed, permissible failure mechanism can be used to estimate the of load-carrying capacity of a simple beam

Example 6.2. Yield Line Method. Continuous beam with cantilever part.

The problem

A continuous beam is simply supported in two locations, however, the beam runs continuously over one of the supports, forming a cantilever part as shown below



Static system of the beam.

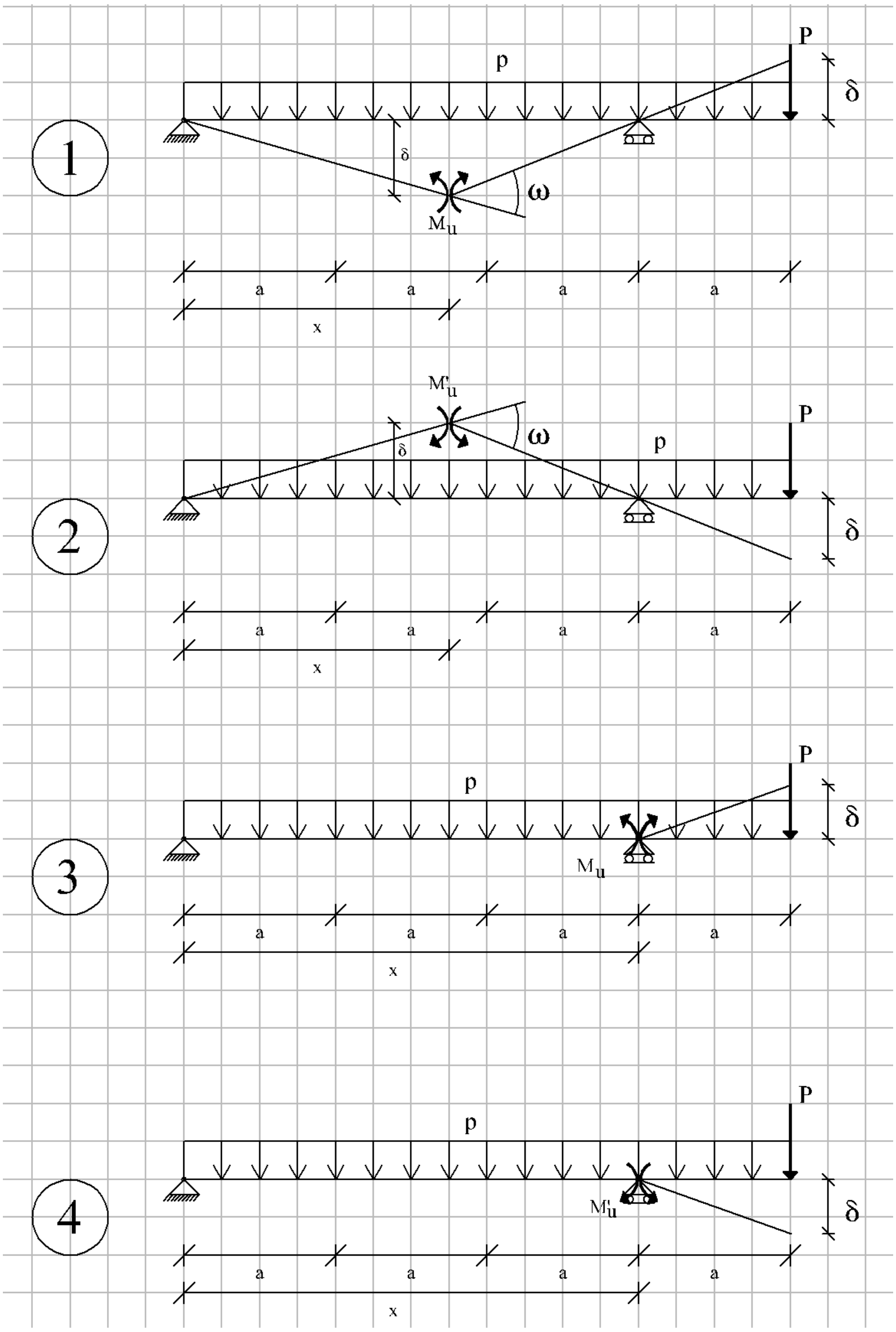
The beam is loaded by a uniform load p and a single load P=pa and has as positive bending moment capacity of M_u and a negative bending moment capacity of $M_u^{'}$, where $M_u^{'} = 2M_u$.

This leads to a system, where the location of the plastic hinge is not so obvious and where we will have to look at several failure mechanisms and optimize these by optimizing the location of the plastic hinge.

Failure mechanism

We notice that the beam must have both positive and negative moments and that we therefore do not know if the plastic hinge will be placed in the part with positive or negative bending, nor do we know if the hinge will be located between the supports or in the cantilever part of the beam.

We must therefore consider a total of four mechanisms:



Calculating the upper limit solution

We must determine the external and internal work for each of these mechanisms and either optimize each of these manually or enter the formulas in eg. a spreadsheet (we could use Excel for this purpose).

Mechanism 1 ($0 \leq x \leq 3a$)

We find that

$$\omega = \frac{\delta}{x} + \frac{\delta}{3a - x} = \delta \frac{3a}{x(3a-x)} \quad \text{and} \quad \delta_P = \frac{\delta}{3a-x} a = \delta \frac{a}{3a-x}$$

after which we may estimate the internal and external work

$$W_i = \omega M_u$$

$$W_e = \frac{\delta}{2} p \cdot 3a - \frac{\delta_P}{2} p \cdot a - \delta_P P = \frac{3}{2}(\delta - \delta_P) pa$$

where $W_i = W_e$ leads to

$$p_1^{(+)} = \frac{\omega M_u}{\frac{3}{2}(\delta - \delta_P)a} = \frac{2M_u}{x(2a-x)} > 0 \text{ which is lowest at x=a, where } p_1^{(+)} = \frac{2}{a^2} M_u$$

Mechanism 2 ($0 \leq x < 3a$)

We notice that this is essentially the opposite mechanism as mechanism 1, except the external work changes sign and M_u is replaced by $M_u^{'}$, so we find

$$p_2^{(+)} = \frac{-\omega M_u^{'}}{\frac{3}{2}(\delta - \delta_P)a} = \frac{-2M_u^{'}}{x(2a-x)} > 0 \text{ which is lowest at x=3a, where } p_2^{(+)} = \frac{4}{3a^2} M_u$$

Mechanism 3 (3a≤x≤4a)

We find that

$$\omega = \frac{\delta}{4a - x}$$

after which we determine the internal and external work and the upper limit solution

$$W_i = \omega M_u$$

$$W_e = -\frac{\delta}{2} p(4a - x) - \delta P = -\frac{\delta}{2} p(6a - x) \Rightarrow$$

$p_3^{(+)} = \dfrac{-2\omega M_u}{\delta(6a - x)} = \dfrac{-2M_u}{(4a - x)(6a - x)} \geq 0$ which is no solution, as it will be negative for all x≥3a.

Mechanism 4 (3a≤x≤4a)

This mechanism is the opposite of mechanism 3, so we just change the sign of the external work and replace the M_u by M_u' and finds

$p_4^{(+)} = \dfrac{2\omega M_u^{'}}{\delta(6a - x)} = \dfrac{2M_u^{'}}{(4a - x)(6a - x)} \geq 0$ which is lowest at x=3a, where $p_4^{(+)} = \dfrac{2}{3a^2} M_u^{'} = \dfrac{4}{3a^2} M_u$

The load-carrying capacity

The loadcarrying capacity is estimated as the lowest, positive value

$$p^{(+)} = \min(p_1^{(+)}, p_2^{(+)}, p_3^{(+)}, p_4^{(+)}) = \frac{4}{3a^2} M_u$$

This corresponds to a hinge, located over the intermediate support – represented by both mechanism 2 and 4 as they agree on the yield hinge placed over the support (in x=3a).

Comments

It should here be noticed that other choices of loads or other choices of M_u and $M_u^{'}$ would lead to other loadcarrying capacities and eventually also to another position of the plastic hinge in the failure mechanism.

The mechanism 3 can't lead to an estimate of the loadcarrying capacity as all p^+-values for any position of the hinge in the x-range between 3a and 4a will be negative – and we could have seen it directly from the drawing of the failure mechanism, as all external loads would either not have any displacement or have a displacement against the direction in which they work.

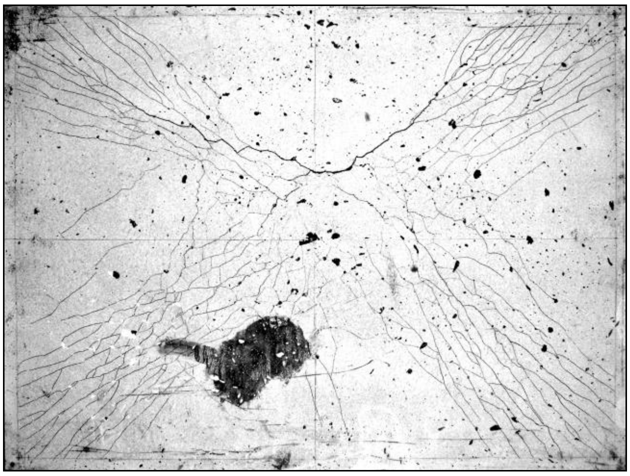
Example 6.3. Yield Line Method.
Rectangular plate with uniform load

The problem

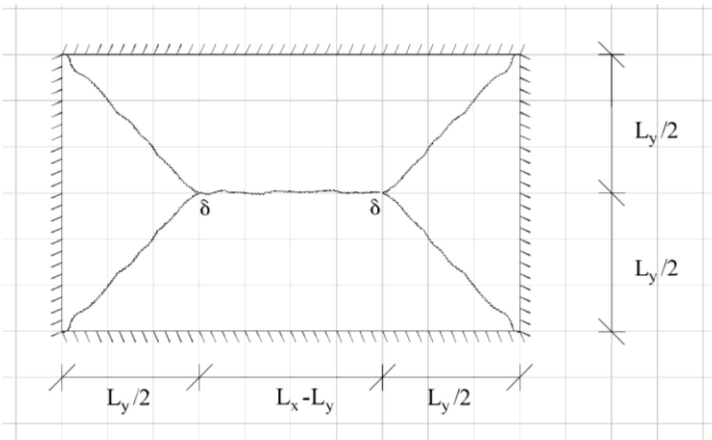
We need to determine the failure mechanism in a rectangular plate, simply supported along all four sides and loaded by a uniform load. The corners as supported in a way, so they can't tilt (lift).

Testing of the slab and failure mechanism

The slab is produced in fibre reinforced concrete and has therefore the same ultimate bending moment capacity m_u in both directions and both sides. It is simply supported by rollers on both sides and along all for sides and is loaded by an airbag.



Failure mechanism the observed failure mechanism shape and the crack patterns.



Failure mechanism

The observed failure mechanism with maximal displacement δ in rectangular plate with uniform load p and simple supports on all 4 sides ($L_y \leq L_x$).

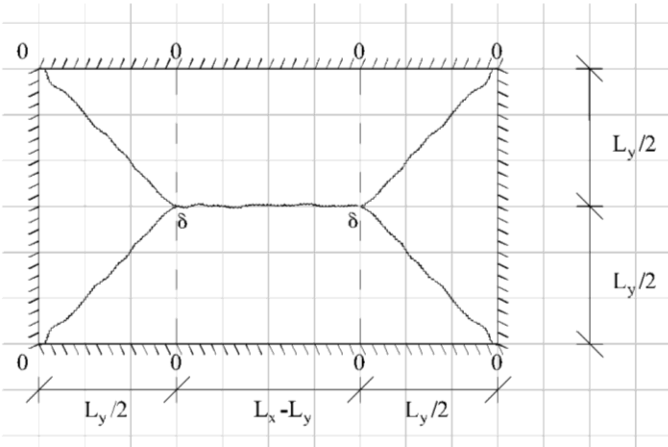
Calculating the upper limit solution

The external work is estimated (as for the beam) as the sum of the loads multiplied with the incremental deformations of the beam in the points, where the loads are applied

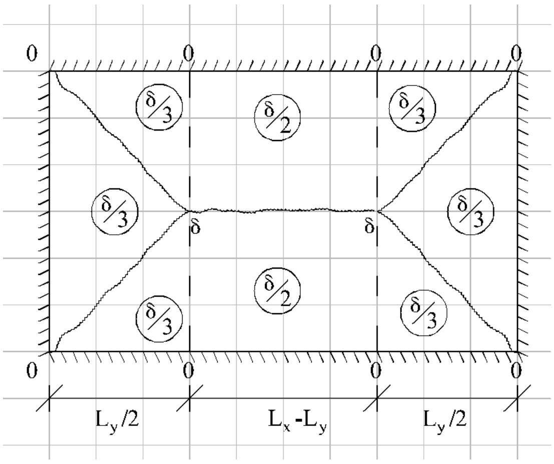
$$W_e = \int p(x,y) \cdot \delta(x,y) \cdot dx \cdot dy + \sum P_i \delta_i$$

This is normally a simple summing up of a number of integral over a number of smaller areas plus a sum of contributions from concentrated loads. It is, however, recommended to go through the following three steps

1. subdivide the failed slab into sections of triangular or rectangular area with a constant load intensity;
2. indicate the average displacement in each section (as the average displacement in a triangle or rectangle is the average of the displacements in the corners of the areas) and then
3. add up the contributions from each section.



Step 1: The failed slab is subdivided into rectangular and triangular sections and the displacements are indicated in all corners



Step 2: The average displacements of each section is indicated in circles.

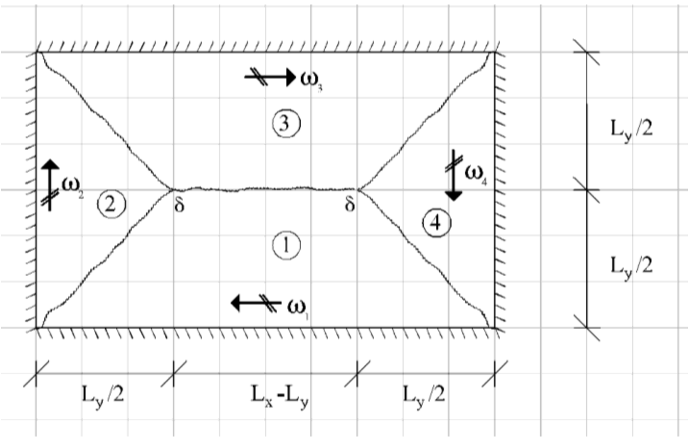
Step 3: The figure shows that the average displacements in the sections are δ/3 or δ/2 in this example, so that the adding up in step 3 leads to

$$\begin{aligned} W_e &= p \cdot (\delta/3) \cdot L_y^{\,2}/2 + p \cdot (\delta/2) \cdot L_y \cdot (L_x - L_y) + p \cdot (\delta/3) \cdot L_y^{\,2}/2 \\ &= p \cdot \delta \cdot L_y \cdot (L_y/3 + (L_x - L_y)/2) \end{aligned}$$

The internal work in the slab is created in the yield lines, just as it was created in the plastic hinge in the beam and depends on the "bend" in the yield line.

The problem with the estimation of the internal work in the yield lines is normally that there is a contribution from each of the yield lines and that the reinforcement directions may not be parallel or perpendicular to the yield line, so the estimation of the internal works in the yield lines may be quite extensive. We would therefore like to simplify the estimations by using the internal work, which is estimated as

$$W_{in} = \int m_u \cdot \omega_{in} \cdot ds = \sum m_u \cdot \omega_{in} \cdot L$$
$$W_i = \sum W_{in}$$



Rotations of the four parts of the slab.

The equivalent, internal work in the slabs section requires a determination of the rotation of the sections, which are determined as

$$\omega_{x1} = 2\delta / b = \omega_{3x}$$
$$\omega_{y2} = 2\delta / b = \omega_{y4}$$

We may now estimate the work in each of the four sections as

$$W_{i1} = m_{ux1}\omega_{y1}L_1 = m_{ux1}\omega_{y1}L_x = m_u 2\frac{\delta}{L_y}L_x = W_{i3}$$

$$W_{i2} = m_{uy2}\omega_{x2}L_2 = m_{uy2}\omega_{x2}L_y = m_u 2\frac{\delta}{L_y}L_y = W_{i4}$$

and finds the total internal work as

$$W_i = W_{i1} + W_{i2} + W_{i3} + W_{i4} = 4m_u(L_x + L_y)\frac{\delta}{L_y}m_u$$

<u>The loadcarrying capacity</u> is now estimated from

$$W_e = W_i$$

which in our example leads to

$$p = \frac{24(L_y + L_x)}{{L_y}^2(3L_x - L_y)} m_u$$

Example 6.4. Yield Line Method.
Quadratic plate with "uniform" load.

The problem

The case will show the development of cracks in the slab and the development of yielding lines in the slab at the higher load levels. The quadratic slab is actually one of the tests, used by K.W. Johansen as documentation for his yield line theory.

Static system of the slab

The slab was intended to be simply supported and loaded by a number of loads, simulating a distributed, uniform load. The slab was therefore placed directly on top of a steel frame, which prevents the slabs sides from moving down, but did not prevent the sides or corners from moving upwards in contrast to our normal, ideal simple support.

We will therefore start by seeing the development of the failure mechanism as reported from the testing, but we will analyse three different cases

1. Simply supported sides with no tilting of the corners (easy);

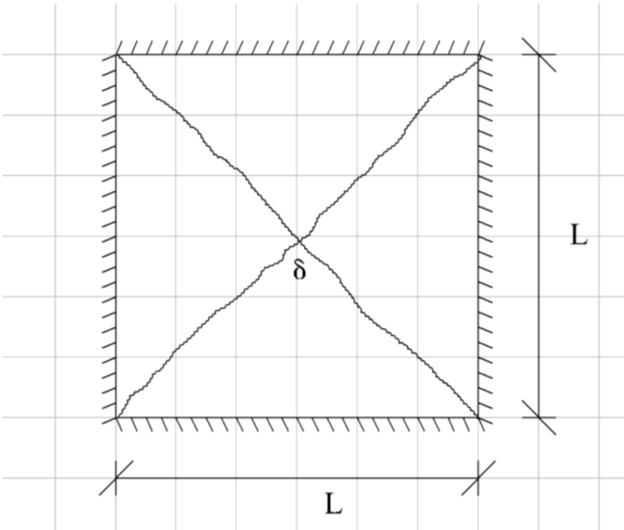
2. Simply supported sides, which allows tilting of the corners (difficult – but KWJ's example);

3. Simply supported sides, with an estimation of the corners forces required to prevent tilting.

Calculating the upper limit solution

Estimating the effect of tilting corners is not difficult, but it does complicate the estimations significantly and we will therefore start with the simple situation, where the quadratic slab is truly simply supported along all four sides with the corners prevented from tilting and the load described as a uniform load p.

Case 1: Quadratic slab with no tilting



The failure mechanism without tilting corners.

In this case we find

$$W_e = 4 \cdot \frac{\delta}{3} p \cdot \frac{1}{2} L \cdot \frac{L}{2} = \frac{1}{3} \delta L$$

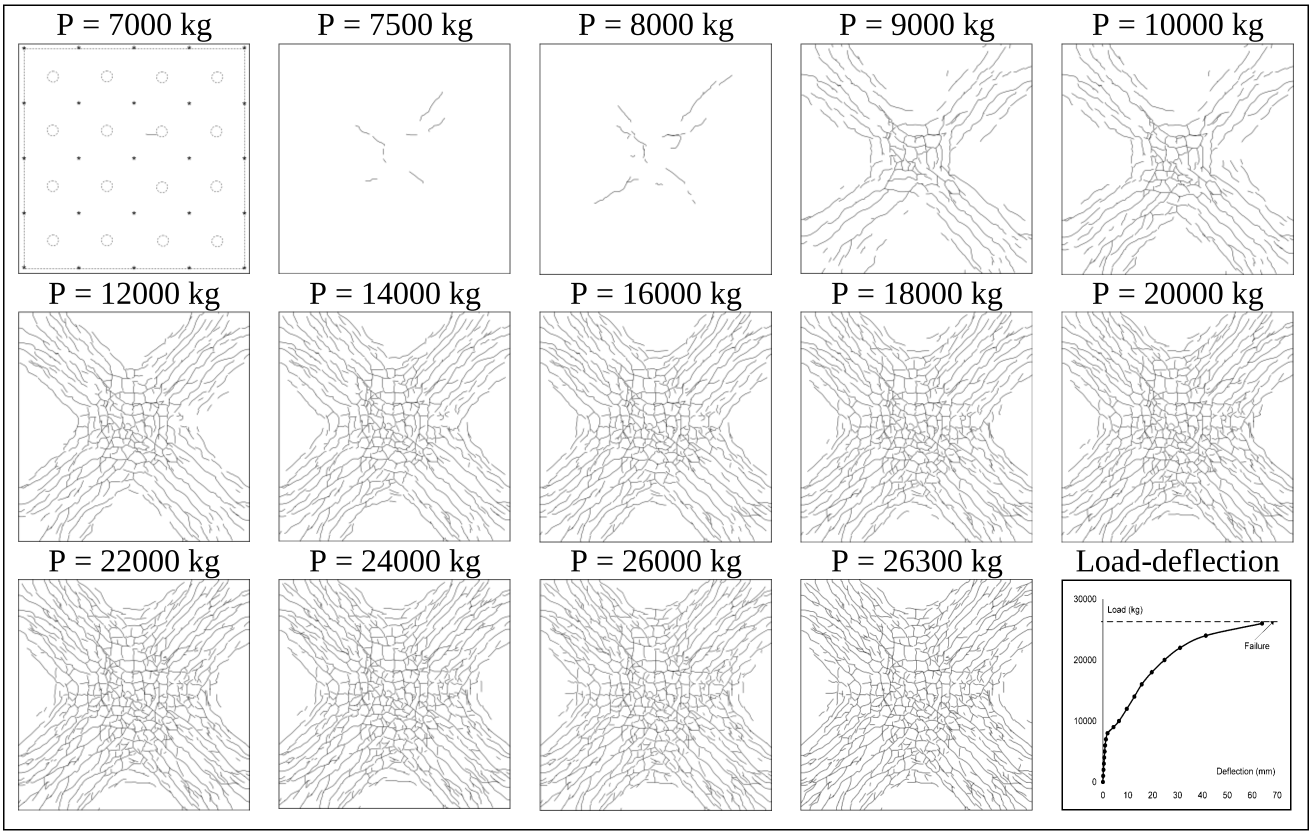
$$W_i = 4\omega m_u L = 4 \frac{\delta}{L/2} m_u L = 8\delta m_u \Rightarrow$$

$$p^+ = 24 \frac{m_u}{L^2}$$

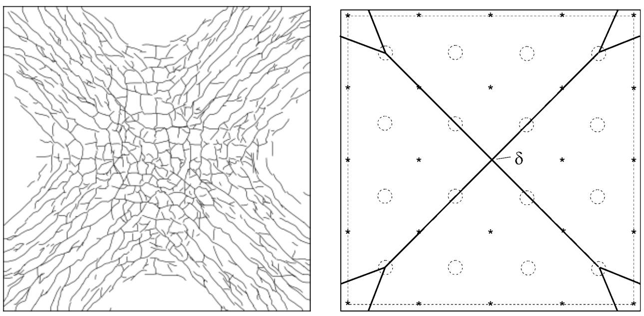
Case 2: Quadratic slab with tilting corners

Testing of the slab.

The failure mechanisms in slabs develop more or less in the same way as the beams plastic hinges, that is, we will observe no cracks at low load levels, then later a few cracks will appear and increase in numbers and widths as the load increases. This was observed in some of the tests shown below.



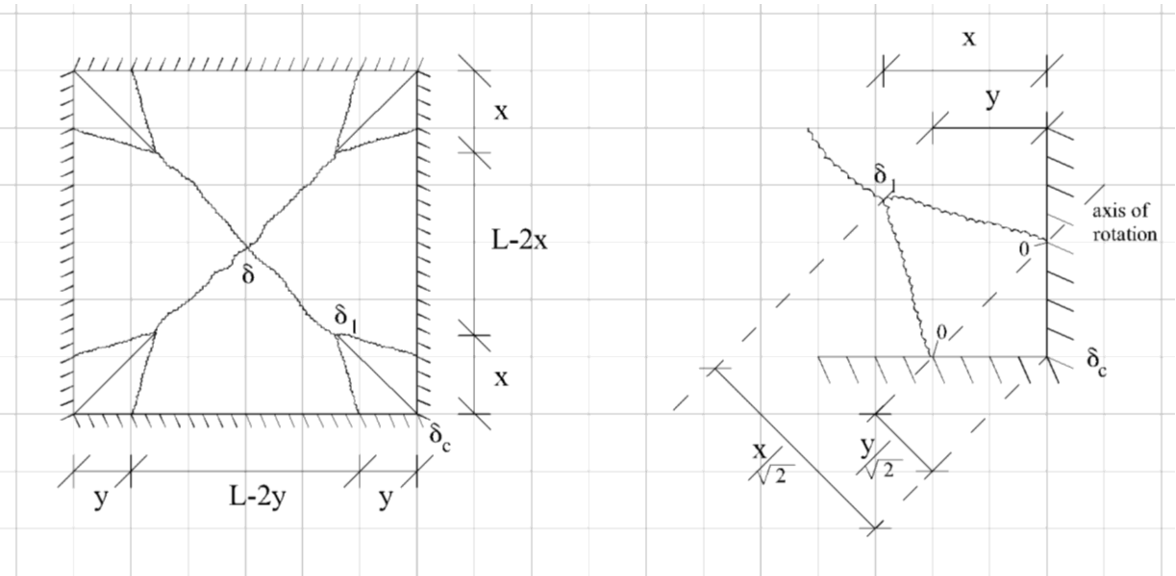
Development of crack patterns at increasing load levels in a square slab (2 x 2 m), supported directly on a frame and loaded in 16 points.



Final crack pattern and observed failure mode.

Estimations

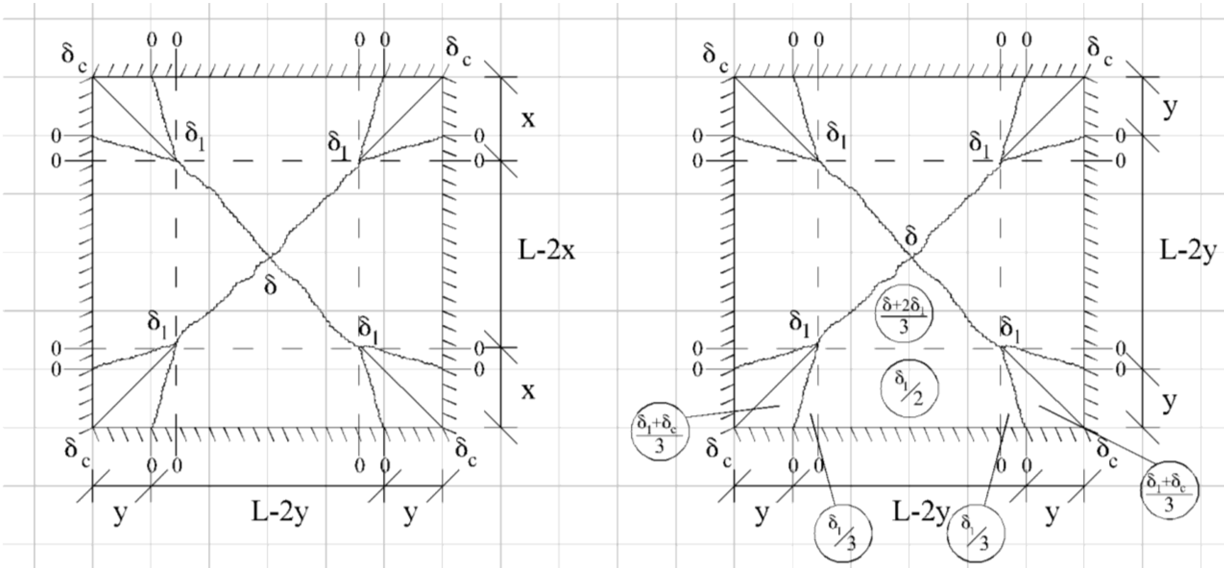
We found that it was easy to estimate the capacity of the square slab without tilting corners was easy and we are now ready to move on to the more complicated situation with the tilting corners, which was tested.



The failure mechanism with tilting corners (left) and details from the corner (right).

We do at the moment not know the distances x and y, but we may estimate the external and internal work as usual, after determining the two last displacements.

$$\delta_1 = \delta \frac{x}{L/2} = 2\delta \frac{x}{L} \text{ and } \delta_c = \frac{-\delta_1}{x\sqrt{2} - y/\sqrt{2}} \cdot y\sqrt{2} = -\delta_1 \frac{y}{2x-y}$$



Division of the slab into triangles and rectangles (left) and indication of the average displacements.

We find now

$$W_e = 4\left[\frac{\delta + 2\delta_1}{3} p \cdot \frac{1}{2}(L-2x)\frac{L-2x}{2} + \frac{\delta_1}{2} p(L-2x)x + 2\frac{\delta_1}{3} p \cdot \frac{1}{2}x(x-y) + \frac{\delta_1 + \delta_c}{3} p(x^2 - 2\cdot\frac{1}{2}x(x-y))\right]$$

$$W_i = 4\left[\omega m_u(L-2y) + \omega_c m_u 2y\right] = 4\left[\frac{\delta}{L/2} m_u(L-2y) + \frac{|\delta_c|}{y} m_u 2y\right] = 8m_u\delta\frac{L-2y}{L} + 8m_u|\delta_c|$$

We can now use the requirement of $W_e = W_i$ to determine the upper limit value of the loadcarrying capacity p^+ for given values of x and y. One may of course try in the traditional way to find the optimal (lowest) value of p^+ by differentiating with respect to x and y, but the engineer would normally use eg. Matlab or Excel to find to this optimal solution, which is

$$p^+ = 22,2\frac{m_u}{L^2}$$
$$x/L = 0,44$$
$$y/L = 0,14 \approx 1/7$$
$$W_i = 0,3197\delta m_u$$
$$W_e = 7,0928\delta pL^2$$

If we compare this to the solution with no tilting of the corners, we find that this mechanism leads to 7,5 % lower capacity p^+, which is not a major difference.

The tests did, however, not use a perfectly uniform load, but loaded the slabs with its own weight and 16 single point loads, which lead to x/L=1/8. Rerunning the optimization of p+ with this x-value we do find

$$y/L = 0,55x/L \Leftrightarrow y/x = 0,55$$

which corresponds very well to the reported crack pattern.

We will normally have the corners restrained, either by the loads from eg. walls or by reinforcement, which anchors the slab to the supports. This reinforcement is normally concentrated in an area around the corner, which covers extend over a length of 1/7 of the shortest side of the slab, just as we found y/L=1/7 in our optimization.

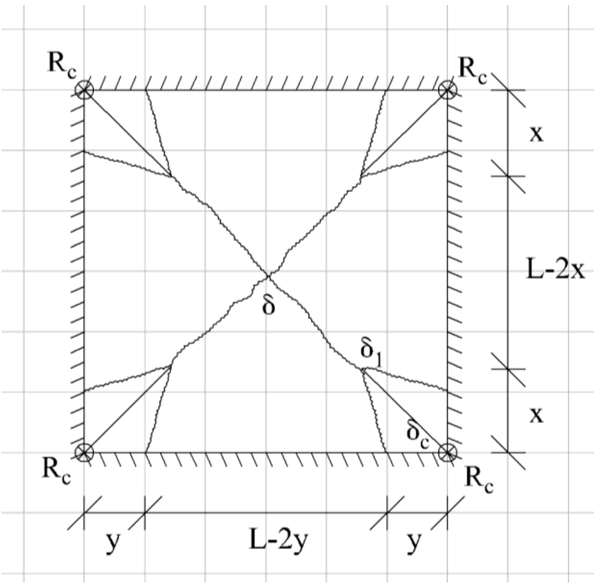
<u>Note:</u> Slabs with simply supported sides and with the corners restrained properly will not be able have tilting corners and this may lead to up to 10 % higher load-carrying capacity according to K.W. Johansen.

Case 3: Quadratic slab with tilting corners with additional loads at the corners

We have now estimated the capacity of the slab without and with tilting corners. We would, however, like to estimate the corner forces required to prevent the tilting, as this force may have to be transferred through additional reinforcement anchored into the supports.

We will estimate this by adding a single load R_c at each corner of the slab, where

$$R_c = npL^2$$



Concentrated reactions at the corners.

We may now estimate the inner work W_i and the external work W_e. These are the same as in the previous situation, except for an additional contribution to W_e as

$$\Delta W_e = 4R_c\delta_c$$

We will start with a very low value of α and estimate the inner and outer work, find the loadcarrying capacity p+ and optimize this with respect to x and y to find the best (lowest) p^+ for this corner reaction.

We notice as expected that the tilt decreases in size with increasing n (as the single loads in the corners should reduce the tilting effect) and we increase n in small steps until y=0 and δ_c =0 is reached. We find

$$n = 0,08333 = 1/12$$

This means that in the case of a quadratic slab with uniform load and the observed yield line pattern, each corner needs to be restrained by a force of $1/12pL^2$ in order to prevent tilting.

References

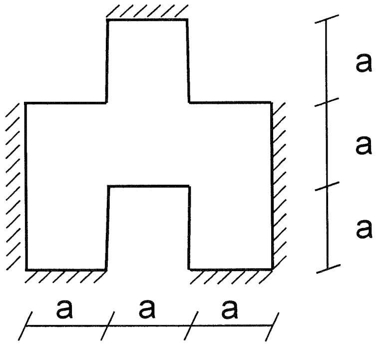
Bach, C. and Graf, O.: “Versuche allseitig aufliegenden, quadratischen und rechteckigen eisenbetonplatten”, Deutscher Ausschuss für Eisenbeton, heft 30, 1915.

Johansen, K. W.: ”Brudlinieteorier”, Jul. Gjellerups Forlag, Copenhagen, 1943, 191 pages. (Yield Line theory, Translated by Cement and Concrete Association, London, 1962, 182 pages).

Example 6.5. Yield Line Method.
Distinguish between possible and impossible mechanisms

The problem

An upper limit solution is needed for the slab shown below. It will require the engineer to determine a few, different failure mechanisms for the yield line methods determination of the load-carrying capacity.

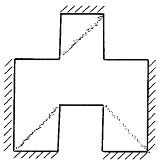
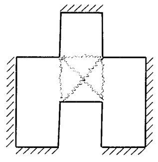
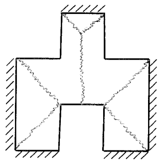
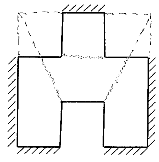
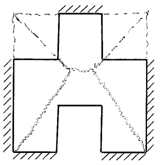
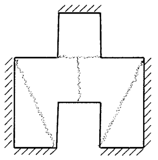
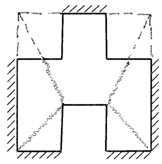
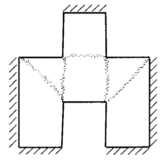
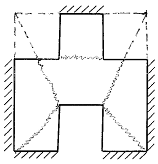
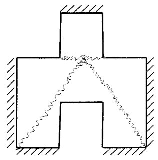
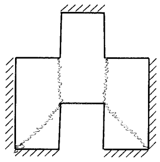
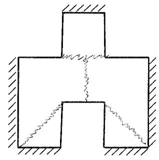


A number (40) of potential yield line patterns have been suggested: Your task is to determine which of the suggested mechanisms are possible and which are impossible.

You will find the correct answers to these questions on the following pages.

An alternative to this is to download the self-quiz, which randomly select a number of yield line patterns from a pool of over 150 figures – you can use the quiz a number of times.

Suggested yield line patterns

Mechanism 1	Mechanism 2	Mechanism 3	Mechanism 4
			
Mechanism 5	Mechanism 6	Mechanism 7	Mechanism 8
			
Mechanism 9	Mechanism 10	Mechanism 11	Mechanism 12
			
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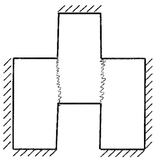	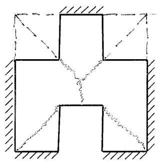	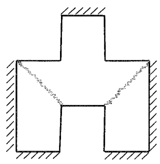	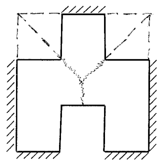
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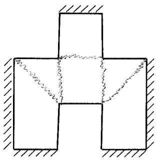	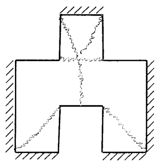	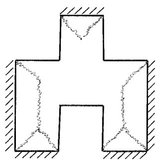	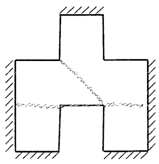
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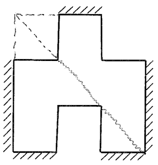	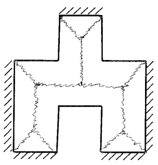	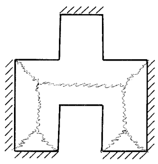	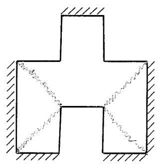

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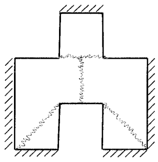	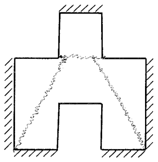	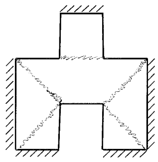	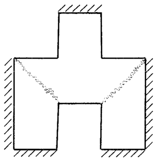
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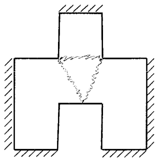	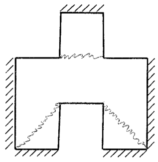	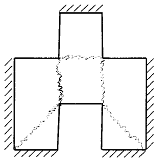	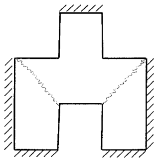
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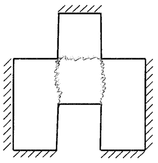	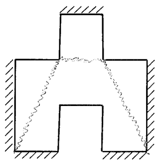	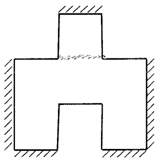	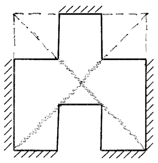
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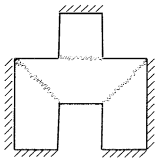
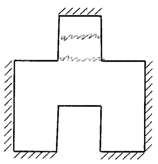
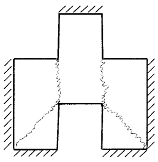
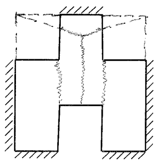
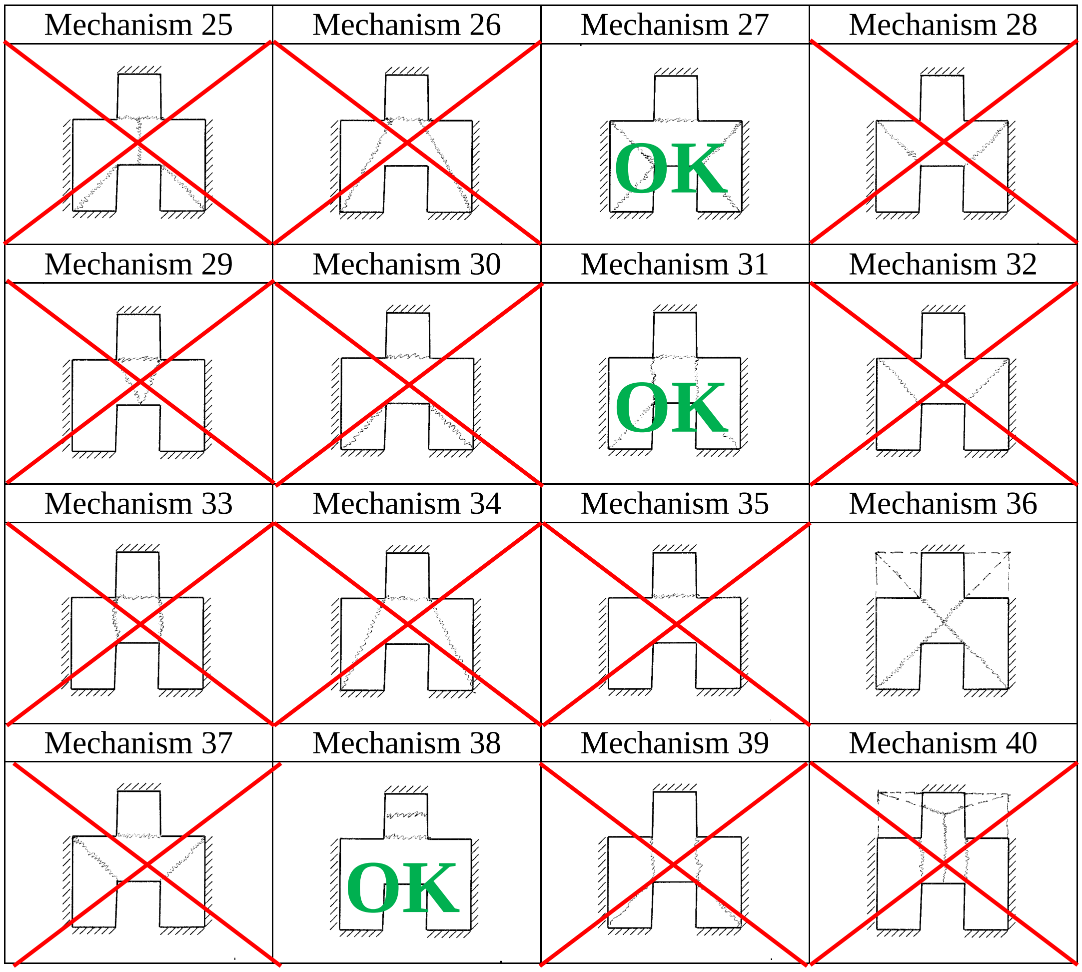
			

Answers to all the mechanisms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